

# Compact convexes of the plane and probability theory

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## Abstract

We revisit the connections between compact convexes of the plane and probability measures. The starting point is a bijection attributed to Gauss-Minkowski, between the set of probability measures  $\mu$  on  $[0, 2\pi]$  such that  $\int_0^{2\pi} e^{ix} d\mu(x) = 0$  and compact convexes of the plane with length 1. We show that some natural operations on convexes – for example, the Minkowski sum – have natural translations in terms of operations on probability measures. Further applications are provided, as a new notion of convolution of convexes, and the proof that a polygonal curve associated with a sample of  $n$  random variables (satisfying  $\int_0^{2\pi} e^{ix} d\mu(x) = 0$ ) converges to a convex associated with  $\mu$  at speed  $\sqrt{n}$ , result much similar to the convergence of empirical process in statistics. In the end, we present some models of smooth random convexes and simulations.

**Keywords :** Random convex, symmetrisation, weak convergence, Minkowski sum.

**AMS classification :** 52A10, 60B05, 60D05, 60F17, 60G99

## 1 Introduction

Convexes are central object in mathematics: they appear everywhere! Busemann [9], Pólya [20] and Pogorelov [19] provide nice overviews on the topic.

In probability theory, convexes appear in 1865 with Sylvester's question [22]. Let  $K$  be the unit square. For  $n = 4$  points chosen independently and at random in  $K$ , what is the probability that these  $n$  points are in a convex position? The same question can be raised for other shapes  $K$ , other values for  $n$ , and other dimensions. Some answers have been provided very recently by Valtr [25, 24] for  $K$  being a triangle or a parallelogram (see also Bárány [3], Buchta [8] and Bárány & al. [2] and references therein).

Random convexes also show up via the studies of Voronoï cells appearing in a Poisson point process (see Calka [10] and references therein), and in the question of the distribution of polygonal convexes (or polygonal convex curves) satisfying some constraints, for example their vertices being attached to some lattice (Sinai [21], Bárány & Vershik [4], Vershik & Zeitouni [26], Bogachev & Zarbaliev [7]). Bárány has many publications on that topic, and we send the interested reader to his papers and web page for additional references.

All these questions and results pertain to convexes that are in fact, polygonal. It seems that more general convexes do not appear in probability theory, at least, random ones. We want here to discuss this, and we want to do this in two absolutely orthogonal directions.

- The first direction can be qualified of fundamental. Theorem 2.1 asserts that the set  $\mathcal{S}$  of probability distributions  $\mu$  with support on  $[0, 2\pi]$ , or more exactly on  $\mathbb{R}/(2\pi\mathbb{Z})$ , satisfying  $\int_0^{2\pi} \exp(ix) d\mu(x) = 0$  (null Fourier transform at time 1), is in one-to-one correspondence with the set of compact convexes of the plane with perimeter 1, considered up to translation; this bijection is in fact an homeomorphism when both sets are equipped with natural topologies. This theorem, proved in Section 2.2, is sometimes called in the literature (e.g. in [26]) the Gauss-Minkowski Theorem; a proof is given in Busemann [9, Section 8]. This property has a lot of immediate and interesting corollaries, some of them being known, but as far as we are aware, they were proved using geometrical arguments, when here we focus on probabilistic ones.

The set  $\mathcal{S}$  is stable by convolution and mixture, and this induces some natural operations on convexes that one may also qualify of *convolution* and *mixture*. It then appears that the mixture of convexes hence defined coincides with the Minkowski addition (Section 3.2). Further, the Minkowski symmetrisation has also a simple interpretation in term of probability theory; it sends a convex associated with a measure  $\mu$ , onto a convex associated with  $\frac{1}{2}(\mu + \mu(2\pi - \cdot))$  (Proposition 3.5). We develop a notion of *convolution of convexes* and *symmetrisation by convolution* (Sections 3.3 and 3.4) which appear to be new, and provide a new proof of the isoperimetric inequality (Theorem 3.7).

Consider now  $\mu \in \mathcal{S}$ , and the curve  $C_n$  formed by i.i.d. increments  $(e^{iX_\mu(j)}, j = 1, \dots, n)$ , where the  $X_\mu(j) \sim \mu$  are sorted according to their arguments and rescaled by  $n$ . We show that the curve  $C_n$  converges when  $n \rightarrow \infty$  to a particular convex  $\mathcal{C}_\mu$  associated with  $\mu$  (Theorem 2.7 and Corollary 2.8). An application to statistics – similar to the convergence of the empirical process toward the Brownian bridge – follows; it states that the curve  $C_n$  converges to  $\mathcal{C}_\mu$  at speed  $\sqrt{n}$ , the difference being described thanks to a Gaussian process (Theorem 2.7).

In the same manner, every distribution of r.v. with support in  $\mathbb{C}$  and mean 0 can be sent on convexes by a second correspondence (which is not bijective) (Section 4.2). Again, the point of view “limit of the curve associated with a sample of  $n$  random variables sorted according to their argument” gives the right intuition.

- The second direction of this paper is the investigation of some models of “random convexes”. The first model we discuss, is again a model of random polygons as follows: take  $z_0, \dots, z_{n-1}$  independent and identically distributed according to a distribution  $\nu$  in  $\mathbb{C}$ . Now, for  $(y_i = z_{i+1 \bmod n} - z_i, 0 \leq i \leq n-1)$ , the  $y_i$ ’s sum to 0, and then up to reordering them according to their arguments, the  $y_i$ ’s are the consecutive vector sides of a convex (for example, the convex with vertices  $(\sum_{j=0}^d y_j, d = 0, \dots, n-1)$ ). We show that when  $n$  goes to  $+\infty$ , this convex converges in distribution, and after rescaling to a deterministic convex (see Theorems 4.2 and 5.1). We

also discuss some questions about the finite case (“ $n$  small”) in Section 5.1.3.

In this work, Fourier transform, and more precisely Fourier series play a great role. For a random variable  $X$  in  $\mathbb{R}$ , and specially for those with support in  $[0, 2\pi]$ , all the Fourier coefficients  $a_n = \mathbb{E}(\cos(nX))$  and  $b_n = \mathbb{E}(\sin(nX))$  are well defined for any  $n \geq 0$ . These coefficients depend on the measure  $\mu$  of  $X$ , and will be denoted  $a_n(\mu), b_n(\mu)$ . Considering the aforementioned bijection between convexes and measures, the question of designing a model of random convexes is equivalent to that of designing a model of random measure  $\mu$  satisfying  $\int_0^{2\pi} \exp(ix) d\mu(x) = 0$ . Clearly, the last equation is equivalent to  $a_1(\mu) = b_1(\mu) = 0$ . But generating random Fourier coefficient  $(a_n, b_n, n \geq 0)$  is not sufficient, since it does not imply the existence of a probability measure  $\mu$  such that  $a_n = a_n(\mu), b_n = b_n(\mu)$  for all  $n$ . In Section 5, we explain how this can be handled, and provide several models of random convexes that are not random polygons.

We would like to end this introduction on a discussion on the real first question we addressed on this topic, even if it is eventually totally absent of this paper. It somehow illustrates the difficulty to design a “natural uniform measure” on random convexes. Take a convex  $C$  of the plane, and consider the maximal convex polyomino with vertices in  $\mathbb{Z}^2$  included in  $C$ . Such a polyomino is called a *digitally convex polyomino* (DCP); an illustration is provided in Figure 1. Let  $D_n$  be the set of DCP with perimeter  $2n$ .

In a recent (unpublished) paper, Bodini, Duchon & Jacquot [6] investigate the limit shape of uniform DCP taken in  $D_n$  under the uniform distribution  $\mathbb{U}_n$ . Even if not convex, these polyominoes can be seen as discretisation of convexes, of all convexes in fact. This model seems to be a good starting point to construct a natural “uniform distribution on convexes”, taking the limit when  $n \rightarrow +\infty$ . But, when  $n$  goes to  $+\infty$ , under  $\mathbb{U}_n$ , polyominoes rescaled by  $n$  converge to a deterministic convex.

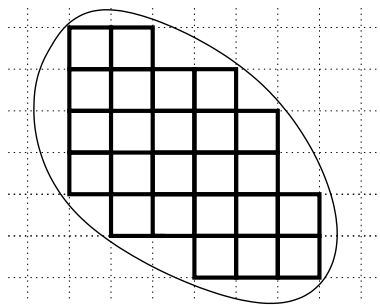


Figure 1: A digitally convex polyomino: it is polyomino composed with the (connected components of the) cells entirely contained in a convex of the plane.

**Convention** The word “convex” will always be used for “compact convex of the plane  $\mathbb{R}^2$ ”. We assume that all the mentioned r.v. are defined on a common probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , and denote by  $\mathbb{E}$  the expectation. For any probability distribution  $\mu$ ,  $X_\mu$  designates a r.v. with

distribution  $\mu$ . We write  $X \sim \mu$  to say that  $X$  has distribution  $\mu$ . The signs  $\xrightarrow[n]{(d)}$ ,  $\xrightarrow[n]{(proba.)}$ ,  $\xrightarrow[n]{(weak)}$  stand for the convergence in distribution, in probability, and the weak convergence.

## 2 Correspondence between convexes and distributions

We start this section by recalling some simple fact concerning convexes and measures on the circle  $\mathbb{R}/(2\pi\mathbb{Z})$ . Thereafter we state the Gauss-Minkowski's theorem (Theorem 2.1) which establishes a correspondence between measures and convexes, and we provide a new proof based on probabilistic arguments. In Section 2.3 we express the area of a convex thanks to the Fourier coefficients of the associated measure. Finally in Section 2.4 we state one of the main result of the paper (Theorem 2.7): under some certain hypothesis, it ensures the convergence of the convex as a trajectory made of  $n$  i.i.d. increments (rescaled by  $n$ ) to a limit convex at speed  $\sqrt{n}$ .

### 2.1 Compact convexes of the plane

A subset  $S$  of  $\mathbb{R}^2$  is *convex* if for any  $z_1, z_2 \in S$ , the segment  $[z_1, z_2] \subset S$ . In this paper, we are interested only in compact convexes of the Euclidean plane  $\mathbb{R}^2$ . Let  $\text{Seg}$  be the set of bounded closed segments, and  $\text{Nei}$  be the set of compact convexes with **non empty interiors**. The union  $\text{Seg} \cup \text{Nei}$  forms the set of all compact convexes of  $\mathbb{R}^2$ .

For convexes  $S \in \text{Nei}$ ,  $S^\circ$  will designate the interior of  $S$ , and  $\partial S = S \setminus S^\circ$  the boundary of  $S$ . We call *parametrisation* of  $\partial S$ , a map  $\gamma : [a, b] \rightarrow \partial S$  for some interval  $[a, b] \subset \mathbb{R}$ , such that  $\gamma(a) = \gamma(b)$  and such that  $\gamma$  is injective from  $[a, b)$  on  $\partial S$  (such a parametrisation exists since one may always send naturally  $[0, 2\pi]$  on  $\partial S$  using a point in  $S^\circ$ ). The length  $|\partial S|$  of  $\partial S$  is well defined, finite and positive. The length of the curve may be used to provide a *natural parametrisation* of  $\partial S$ , that is a function  $\gamma : [0, |\partial S|] \rightarrow \partial S$ , continuous and injective on  $[0, |\partial S|)$ , such that  $\gamma(0) = \gamma(|\partial S|)$  and such that the length of  $\{\gamma(t), t \in [0, s]\}$  is  $s$  for any  $s \in [0, |\partial S|]$ . For convexes  $S \in \text{Seg}$ , the notion of natural parametrisation also exists, but is different. For technical reasons, we choose the following one : The natural parametrisation of a segment  $[a, b]$  is defined to be  $\gamma(t) = a(1 - \frac{t}{|b-a|}) + b\frac{t}{|b-a|}$  on  $[0, |b-a|]$  and  $\gamma(t) = a(\frac{t}{|b-a|} - 1) + b(2 - \frac{t}{|b-a|})$  on  $[|b-a|, 2|b-a|]$ , as if the segments were thick and two-sided.

In the sequel, we will often call convex the border of a convex.

### 2.2 Measures on the circle

Let  $\mathcal{T}$  be the circle  $\mathbb{R}/(2\pi\mathbb{Z})$  equipped with the quotient topology. The set of probability distributions  $\mathcal{M}_{\mathcal{T}}$  on  $\mathcal{T}$  is different from  $\mathcal{M}_{[0, 2\pi]}$  and also different from  $\mathcal{M}_{[0, 2\pi)}$ , due to  $\delta_0$ , the Dirac mass at 0 whose neighbourhood is different in these sets. The weak convergence on  $\mathcal{M}_{\mathcal{T}}$  is defined as usual :  $(\mu_n, n \geq 0) \xrightarrow[n]{(weak)} \mu$  in  $\mathcal{M}_{\mathcal{T}}$  if for any bounded continuous function  $f : \mathcal{T} \rightarrow \mathbb{R}$ ,

$\int_{\mathcal{T}} f d\mu_n \rightarrow \int_{\mathcal{T}} f d\mu$ . Let

$$\begin{aligned} F_\mu : \mathcal{T} &\longrightarrow [0, 1] \\ x &\longmapsto \mu([0, x]) \end{aligned}$$

be the cumulative distribution function (CDF) of  $\mu \in \mathcal{M}_{[0, 2\pi]}$ . Let  $\mathcal{I}_\mu$  be the set of points of continuity of  $F_\mu$ , where by convention,  $0 \in \mathcal{I}_\mu$  if  $F_\mu(0) = \mu(\{0\}) = 0$ . If  $\mu(n) \xrightarrow[n]{(weak)} \mu$  in  $\mathcal{M}_{\mathcal{T}}$ , then it can not be deduced that  $F_{\mu(n)} \rightarrow F_\mu$  pointwise on  $\mathcal{I}_\mu$  since  $\delta_{2\pi} = \delta_0$  in  $\mathcal{M}_{\mathcal{T}}$ . What is still true, is that

$$F_{\mu(n)}(y) - F_{\mu(n)}(x) \rightarrow F_\mu(y) - F_\mu(x), \text{ for any } (x, y) \in \mathcal{I}_\mu.$$

A function  $F : [0, 2\pi) \rightarrow \mathbb{R}$  is a CDF of some distribution  $\mu \in \mathcal{M}_{\mathcal{T}}$  if it is right continuous, non decreasing on  $[0, 2\pi]$ , satisfies  $0 \leq F(0) \leq 1$ ,  $F(2\pi-) = 1$  (in fact, if we had worked in  $[0, 2\pi]$  instead, we would have required  $F(2\pi) = 1 + F(0)$ ). See Wilms [27, p.4-5] for additional information and references.

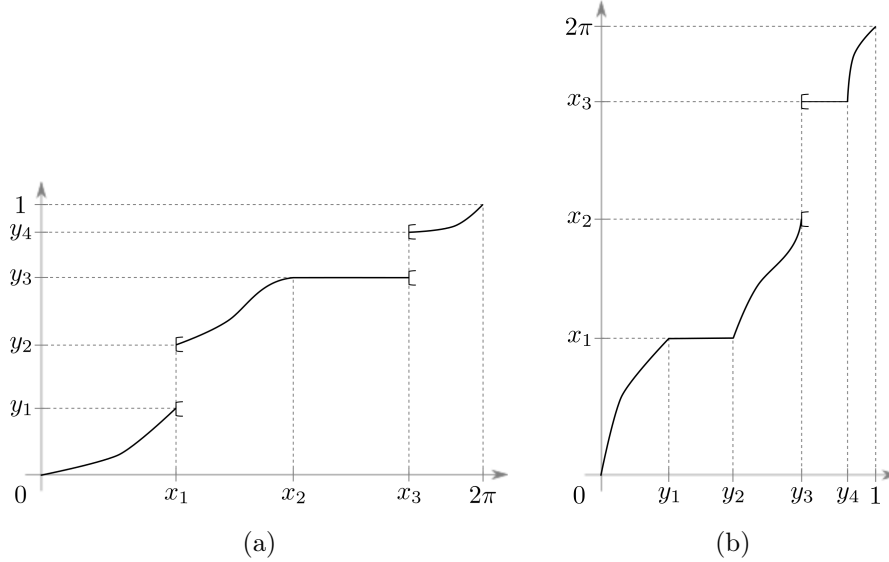


Figure 2: (a) A cumulative distribution function  $F$ , and (b) its generalised inverse  $F^{-1}$ .

Consider the function

$$\begin{aligned} Z_\mu : [0, 1] &\longrightarrow \mathbb{C} \\ t &\longmapsto Z_\mu(t) = \int_0^t \exp(iF_\mu^{-1}(u)) du, \end{aligned} \tag{1}$$

where  $F_\mu^{-1} : [0, 1] \rightarrow [0, 2\pi)$  is the generalised inverse distribution function of  $\mu$ , defined by

$$F_\mu^{-1}(y) := \inf\{x \geq 0 : F_\mu(x) \geq y\},$$

and illustrated in Fig. 2. The range  $\mathcal{C}_\mu$  of  $Z_\mu$  is the central object here:

$$\mathcal{C}_\mu := \{Z_\mu(t), t \in [0, 1]\}.$$

Since  $Z_\mu$  has a derivative with modulus 1, it is the natural parametrisation of  $\mathcal{C}_\mu$ , and then  $\mathcal{C}_\mu$  has length 1.

Let  $\mathbf{Conv}$  be the set of compact convexes of the plane containing the point  $(0,0)$ , lying above the  $x$ -axis, and whose intersection with the  $x$ -axis is included in  $\mathbb{R}^+$ . Denote by  $\mathbf{Conv}(1)$  be the subset of  $\mathbf{Conv}$  of convexes having length 1. Set

$$\mathcal{M}_{\mathcal{T}}^0 = \left\{ \mu \in \mathcal{M}[0, 2\pi] , \int_0^{2\pi-} \exp(i\theta) dF_\mu(\theta) = 0 \right\}.$$

be the subset of  $\mathcal{M}_{\mathcal{T}}$  of measures having Fourier transform equal to 0 at time 1 (recall that the Fourier transform of  $\mu$  is  $\Psi_\mu : t \mapsto \mathbb{E}(\exp(itX_\mu))$ .)

Probability distributions on  $\mathbb{R}$  are characterised by their Fourier transform, and the convergence of the Fourier transforms characterises the weak convergence (this is Lévy's famous continuity Theorem). The following Theorem gives a similar characterisation of measures in  $\mathcal{M}_{\mathcal{T}}^0$  by their representation as convexes of the plane. Moreover, this characterisation is an homeomorphism and therefore preserves the notion of convergence in each world.

**Theorem 2.1.** 1) *The function*

$$\begin{aligned} \mathcal{C} : \mathcal{M}_{\mathcal{T}}^0 &\longrightarrow \mathbf{Conv}(1) \\ \mu &\longmapsto \mathcal{C}_\mu \end{aligned}$$

*is a bijection.*

2)  $\mathcal{C}$  is an homeomorphism from  $\mathcal{M}_{\mathcal{T}}^0$  (equipped with the weak convergence topology) to  $\mathbf{Conv}(1)$  (equipped with the Hausdorff topology on compact sets).

This Theorem sometimes called “Gauss-Minkowski” in the literature can be found in a slightly different form in Busemann [9, Section 8]. There, this theorem is stated more generally in  $\mathbb{R}^n$ , where the measures range over the unit sphere of  $\mathbb{R}^n$  and verify a set of properties, which in  $\mathbb{R}^2$  sum up to  $\int_0^{2\pi} e^{ix} d\mu(x) = 0$ . The formula (1) based on  $F^{-1}$ , which is central in our setting, does not seem to appear in the literature. We provide a proof of Theorem 2.1 in probabilistic terms at the end of this section.

**Remark 2.2.** The map  $\mathcal{C}$  that one may qualify of “curve” transform, may be extended to  $\mathcal{M}[0, 2\pi]$ ; in this case  $\mathcal{C}(\mathcal{M}[0, 2\pi])$  is the set of continuous almost everywhere differentiable curves of length 1, starting at  $(0,0)$ , having a positive argument in a neighbourhood of 0, and where along an injective parametrisation, the argument of the tangent is increasing. These curves contain at most one self-intersection point<sup>1</sup>.

Recall that if  $U \sim \text{uniform}[0, 1]$  then  $F_\mu^{-1}(U) \sim \mu$ , and then

$$Z_\mu(t) = \mathbb{E} \left( \mathbf{1}_{U \leq t} \exp(iF_\mu^{-1}(U)) \right). \quad (2)$$

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<sup>1</sup>The Fourier transform  $t \mapsto \Psi_\mu(t)$  also defines a curve  $\{\Psi_\mu(t) : t \in A\}$  in the plane, for any interval  $A$ . This curve is different from  $\mathcal{C}_\mu$ , for any  $A$ .

If  $\mathbf{1}_{U \leq t} = \mathbf{1}_{F_\mu^{-1}(U) \leq F_\mu^{-1}(t)}$  then we get directly Formula (4), but in general, this is false since  $F_\mu^{-1}$  is not strictly increasing. But,  $\{u : u \leq t\} = \{u : F_\mu^{-1}(u) \leq F_\mu^{-1}(t)\}$  if and only if  $t = 1$  or if  $F_\mu^{-1}$  is increasing at the right of  $t$  (meaning that for any  $v > t$ ,  $F_\mu^{-1}(v) > F_\mu^{-1}(t)$ ). Let

$$I_\mu := \{t : F_\mu^{-1}(t^+) > F_\mu^{-1}(t^-)\} \cup \{1\}. \quad (3)$$

In short,  $I_\mu$  represents  $[0, 1]$  deprived from the interiors of the intervals where  $F^{-1}$  is constant. Equivalently,  $I_\mu = \overline{\{F_\mu(t), t \in [0, 2\pi]\}} \cup \{1\}$ . For any  $t \in I_\mu$ , we have from (2),

$$Z_\mu(t) = \mathbb{E} \left( \mathbf{1}_{X_\mu \leq F_\mu^{-1}(t)} \exp(iX_\mu) \right). \quad (4)$$

Since the property  $x \leq F_\mu^{-1}(y)$  is equivalent to  $F_\mu(x) \leq y$ , then for any  $t \in I_\mu$ ,

$$Z_\mu(t) = \mathbb{E} \left( \mathbf{1}_{F_\mu(X) \leq t} \exp(iX_\mu) \right). \quad (5)$$

From (2), we see that  $Z_\mu$  is linear on any interval composing the complement of  $I_\mu$  in  $[0, 1]$ : if  $(t_1, t_2)$  is such an interval, for any  $t \in [t_1, t_2]$ ,

$$Z_\mu(t) = Z_\mu(t_1) + (t_2 - t) \frac{Z_\mu(t_2) - Z_\mu(t_1)}{t_2 - t_1}.$$

Now, an *extremal point* of a convex  $C$  is a point  $x \in C$  such that if  $x = \lambda a + (1 - \lambda)b$  for some  $a, b \in C$ , and  $\lambda \in [0, 1]$ , then  $a = b = x$ . Let  $\text{Ext}(C)$  be the set of extremal points of  $C$ .

**Lemma 2.3.** *For any  $\nu \in \mathcal{M}_T^0$ ,*

$$\text{Ext}(C_\nu) = \overline{\{Z_\nu(F_\nu(t)), t \in [0, 2\pi]\}}.$$

**Proof.** By what is said above, the points in the complement of  $I_\nu$  are not extremal, and reciprocally, every non-extremal point lies on a segment inside the border of  $C_\nu$  and necessarily belongs to the complement of  $I_\nu$ . Therefore  $\text{Ext}(C_\nu) = \{Z_\nu(t), t \in I_\nu\}$ . And following the definition of  $I_\nu$ ,  $I_\nu = \overline{\{F_\nu(t), t \in [0, 2\pi]\}}$ . The final results stems from the fact that  $Z_\nu^{-1}$  is continuous, since  $Z_\mu$  is an isometry from  $\mathbb{R}/\mathbb{Z}$  into  $C_\mu$ .  $\square$

The curvature  $k_\mu(t)$  of  $C_\mu$  at time  $t$ , is given by  $\frac{1}{F'_\mu(F_\mu^{-1}(t))}$  when  $F_\mu$  admits a derivative at  $F_\mu^{-1}(t)$ ; in particular, this means that when  $\mu$  admits a density  $f_\mu$ , then  $k_\mu(F_\mu(\theta)) = 1/f_\mu(F^{-1}(F(\theta))) = 1/f_\mu(\theta)$  (this is the curvature where the argument of the tangent is  $\theta$ ).

The real and imaginary parts  $x_\mu(t) = \Re(Z_\mu(t))$  and  $y_\mu(t) = \Im(Z_\mu(t))$  of  $Z_\mu(t)$  satisfy

$$\begin{cases} x_\mu(t) &= \int_0^t \cos(F_\mu^{-1}(u)) du = \int_0^{F_\mu^{-1}(t)} \cos(v) dF(v) \\ y_\mu(t) &= \int_0^t \sin(F_\mu^{-1}(u)) du = \int_0^{F_\mu^{-1}(t)} \sin(v) dF(v). \end{cases}$$

the second equality in each line being valid only for  $t \in I_\mu$ .

**Proof of Theorem 2.1(1) .** a) First, we prove that for any  $\mu \in \mathcal{M}_T^0$ ,  $C_\mu$  is a compact convex. The function  $Z_\mu$  is continuous. A simple analysis shows that  $y_\mu$  is a continuous function such

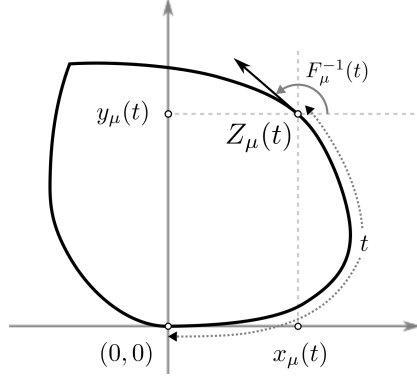


Figure 3: A convex  $\mathcal{C}_\mu$  for some measure  $\mu$ ,  $t$  gives the length of the curve between 0 and  $Z_\mu(t)$  (in the trigonometric order),  $F_\mu^{-1}(t)$  is then the direction of the tangent at time  $t$

that  $y_\mu(0) = y_\mu(1) = 0$ , and that is increasing then decreasing over  $[0, 1]$ . Therefore,  $\mathcal{C}_\mu$  lies on the half plane above the  $x$ -axis. More generally, for any  $\theta \in [0, 2\pi)$ ,  $\mu_\theta(\cdot) = \mu(\cdot - \theta \bmod 2\pi)$  is still in  $\mathcal{M}_T^0$ , and  $\mathcal{C}_{\mu_\theta}$  lies on the half plane above the  $x$ -axis. Therefore, for all  $t \in [0, 1]$ , the line  $D_t$  passing through  $Z_\mu(t)$  making an angle  $F_\mu^{-1}(t)$  with the origin, is called a *supporting line* of  $\mathcal{C}_\mu$ . Since  $F_\mu^{-1}$  is right-continuous,  $\mathcal{C}_\mu$  is tangent to  $D_t$ , and lies entirely in the half-plane separated by  $D_t$  that corresponds to the angles  $[F_\mu^{-1}(t), F_\mu^{-1}(t) + \pi]$ .

We can now show that  $\mathcal{C}_\mu$  is a simple curve or a segment : let  $z$  be such that  $z = Z_\mu(t_1) = Z_\mu(t_2)$ , for  $t_1 < t_2$ . Considering (1), this says that  $\int_{t_1}^{t_2} \exp(iF_\mu^{-1}(u)) du = 0$  and viewing this as a weighted barycentre of a portion of the circle, this implies that  $F^{-1}(t_2) - F^{-1}(t_1) \geq \pi$  and  $2\pi - F^{-1}(t_2) + F^{-1}(t_1) \geq \pi$ . Finally, this implies  $F^{-1}(t_2) = \pi + F^{-1}(t_1) \bmod 2\pi$ , and  $\mu(\{t_2\}) = \mu(\{t_1\}) = 1/2$ , in other words, the convex is a segment of length 1/2. Therefore, when  $\mathcal{C}_\mu$  is not a segment, it is a bounded Jordan curve, that encloses a bounded connected subset  $B_\mu$ .

Let us prove now that  $\mathcal{B}_\mu$  is a convex subset of the plane, by showing that if  $z_1 = Z_\mu(t_1)$  and  $z_2 = Z_\mu(t_2)$  belong to  $\mathcal{C}_\mu$  for  $t_1 < t_2$ , then  $[z_1, z_2] \subset B_\mu$ . Up to a multiplicative and an additive constant, we can redefine  $Z_\mu$  so that  $[z_1, z_2]$  is situated on the horizontal axis, and that  $F_\mu^{-1}(t_1) \in [0, \pi]$  so that the curve is aimed at the upper half-plane at  $t_1$ . Then remark that  $F_\mu^{-1}(t_2) \in [\pi, 2\pi]$ , so that the point  $z_1$  is situated in the correct half-plane separated by  $D_{t_2}$ . The function  $t \mapsto \Im(Z_\mu(t))$  is equal to  $t \mapsto \int_{t_1}^t \sin(F_\mu^{-1}(u) + \theta) du$  where  $\theta$  is a constant, and is increasing then decreasing over  $t \in [t_1, t_2]$ . Therefore,  $Z_\mu([t_1, t_2])$  is situated in the upper half-plane. A symmetric argument shows that  $Z_\mu([0, t_1] \cup [t_2, 1])$  is situated in the lower half-plane. Therefore,  $[z_1, z_2] \subset B_\mu$ .

b) The injectivity of  $\mathcal{C}$  is clear since if  $F_\mu^{-1}(t) = F_\nu^{-1}(t)$  for all  $t \in [0, 1]$ , then  $\mu = \nu$ . Now, let  $C$  be a convex in  $\text{Conv}(1)$  and consider the unique natural parametrisation  $Z$  of  $C$  such that  $Z(0) = Z(1) = (0, 0)$  and such that  $\arg Z(0^+) > 0$  (that is,  $C$  is parametrised following the trigonometric orientation). The map  $Z$  has a.e. a derivative  $g$ , and since it is continuous,



$g$  is the derivative of  $Z$  in the distribution sense:  $Z(t) = \int_0^t g(s)ds$ . Now, one may consider that  $g$  is the natural parametrisation of  $C$  which leads  $g(s) = \exp(iG(s))$  for some function  $G : [0, 1] \rightarrow [0, 2\pi)$ , non decreasing. Hence  $G$  has a right continuous modification  $\tilde{G}$  which also satisfies  $Z(t) = \int_0^t e^{i\tilde{G}(s)}ds$ . The function  $\tilde{G}$  is the inverse of a CDF  $F_\nu$  for some  $\nu$  in  $\mathcal{M}_\mathcal{T}^0$ .

**Proof of Theorem 2.1 (2)** . Consider first the continuity of  $\mathcal{C}$ . For any  $t \in [0, 2\pi)$  and any pair of distributions  $(\mu, \nu)$ , since  $x \rightarrow \exp(ix)$  is 1-Lipschitz,

$$\begin{aligned} |Z_\mu(t) - Z_\nu(t)| &= \left| \int_0^t \exp(iF_\mu^{-1}(u)) - \exp(iF_\nu^{-1}(u))du \right| \\ &\leq \int_0^t d_\mathcal{T}(F_\mu^{-1}(u), F_\nu^{-1}(u))du, \end{aligned}$$

where  $d_\mathcal{T}$  is the distance in  $\mathcal{T}$ , defined for  $0 \leq x \leq y < 2\pi$  by  $d_\mathcal{T}(x, y) = \min\{y - x, 2\pi - y + x\}$ . This last quantity is then bounded above, uniformly in  $t \in [0, 1]$  by  $\mathbb{E}(d_\mathcal{T}(X_\mu, X_\nu))$ , for

$$X_\mu := F_\mu^{-1}(U), \quad X_\nu := F_\nu^{-1}(U),$$

where  $U \sim \text{uniform}[0, 2\pi]$ . Now,  $\mathbb{E}(d_\mathcal{T}(X_\mu, X_\nu))$  is a Wasserstein like distance  $W_1(\mu, \nu)$  between the distributions  $\mu$  and  $\nu$  in  $\mathcal{T}$  (in the standard Wasserstein distance, the circle condition is absent). Now, it is classical that the convergence in distribution implies the convergence of the Wasserstein distance to 0. This property can be easily extended in the present case (using, for example, that  $X_n \xrightarrow[n]{(d)} X$  in  $\mathcal{M}_\mathcal{T}$  iff there exists  $\theta \in [0, 2\pi]$  (any point of continuity of  $X$  does the job) for which  $X_n - \theta \bmod 2\pi \xrightarrow[n]{(d)} X - \theta \bmod 2\pi$  in the standard sense).

Reciprocally, let  $(C_n, n \geq 0)$  be a sequence of convexes  $C_n$  converging to  $\mathcal{C}_\mu$  for the Hausdorff distance  $(d_H)$ . By Theorem 2.1(1), there exists  $\mu_n \in \mathcal{M}_\mathcal{T}^0$  such that  $\mathcal{C}_{\mu_n} = C_n$ . We now establish that  $(\mu_n, n \geq 0)$  possesses exactly one accumulation point, equal to  $\mu$ . Consider a subsequence  $F_{\mu_{n_k}}$  such that  $F_{\mu_{n_k}} \xrightarrow{D_1} G$ , where  $G$  is the repartition function of a measure  $\nu$ . Such a subsequence exists by an argument of compactness. Now :  $F_{\mu_{n_k}} \xrightarrow{D_1} G \Rightarrow F_{\mu_{n_k}}^{-1} \xrightarrow{D_1} G^{-1}$ . According to the first part of this proof, the limit convex  $\mathcal{C}_\nu$  must be equal to  $\mathcal{C}_\mu$ . Since by Theorem 2.1(1), the convexes characterise the measures,  $\nu \stackrel{(d)}{=} \mu$ .  $\square$

A *supporting half-plane* for  $\mathcal{C}_\mu$  is a closed half-plane  $H$  such that  $H \cap \mathcal{C}_\mu = \mathcal{C}_\mu$  and  $\partial H \cap \mathcal{C}_\mu \neq \emptyset$ .

**Lemma 2.4.**  $\mathcal{C}_\mu$  is equal to the intersection of its supporting half-planes.

**Proof.** Clearly,  $\mathcal{C}_\mu$  is included in the intersection of its supporting half-planes. Reciprocally, let  $x$  be a point outside the convex, and  $y$  a point in  $\mathcal{C}_\mu$  such that  $d(x, y) = d(x, \mathcal{C}_\mu)$ . Let  $T$  be the tangent to  $\text{Circle}(x, d(x, y))$  at  $y$ .  $T$  separates the plane into two half-planes, one (open) containing  $x$  and the other  $\mathcal{C}_\mu$ .  $\square$

## 2.3 Fourier decomposition of the convex curve

Fourier coefficients play a central role in the present paper, as they provide powerful tools to manipulate the measures in  $\mathcal{M}_\mathcal{T}^0$  : first, some geometric features of the convexes  $\mathcal{C}_\mu$ , such as their

area, have an elegant expression with the Fourier coefficients of  $\mu$  (defined below); the second reason that will appear later on in the paper, is that the Fourier coefficient of the measures provide a natural way to represent the measures satisfying  $\int_0^{2\pi} e^{ix} d\mu(x)$ , thus allowing the study of models of random convexes.

Let  $f$  be a map from  $[0, 2\pi]$  with values in  $\mathbb{R}$ . Set, for any  $k \geq 0$ ,

$$a_k = \pi^{-1} \int_0^{2\pi} \cos(ku) f(u) du, \quad b_k = \pi^{-1} \int_0^{2\pi} \sin(ku) f(u) du.$$

The quantity  $\frac{1}{2}a_0 + \sum_{k \geq 1} a_k \cos(ku) + b_k \sin(ku)$  is the standard Fourier series of  $f$ . For  $\mu$  in  $\mathcal{M}_{\mathcal{T}}$  (or in  $\mathcal{M}[0, 2\pi]$ ), the Fourier coefficients of  $\mu$  are defined, for any  $k \geq 0$  by

$$a_k(\mu) = \pi^{-1} \mathbb{E}(\cos(kX_\mu)), \quad b_k(\mu) = \pi^{-1} \mathbb{E}(\sin(kX_\mu)). \quad (6)$$

Hence,  $a_0(\mu) = 1/\pi$ . Of course, the condition  $\int_0^{2\pi} e^{iu} dF_\mu(u) = 0$  coincides with

$$a_1(\mu) = \mathbb{E}(\cos(X_\mu)) = 0, \quad b_1(\mu) = \mathbb{E}(\sin(X_\mu)) = 0. \quad (7)$$

The following Proposition, whose proof can be found in Wilms [27, Theorem 1.6 and 1.7], says that probability measures are characterised by their Fourier coefficients, and establishes a continuity theorem.

**Proposition 2.5.** (1) *The function*

$$\begin{aligned} \text{Coeffs} : \mathcal{M}_{\mathcal{T}} &\longrightarrow \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \\ \mu &\longmapsto ((a_k(\mu), k \geq 0), (b_k(\mu), k \geq 1)) \end{aligned}$$

*is injective.*

(2) *Let  $\mu, \mu_1, \mu_2, \dots$  be a sequence of measures in  $\mathcal{M}$ . The two following statements are equivalent:  $\mu_n \xrightarrow[n]{(weak)} \mu$  and  $\text{Coeffs}(\mu_n) \rightarrow \text{Coeffs}(\mu)$  (for the simple convergence).*

**Example 2.6.** – If  $\mu \sim \text{uniform}[0, 2\pi]$  then  $a_k(\mu) = b_k(\mu) = 0$  for any  $k \geq 1$ .

– If  $\mu = \sum_{k=0}^{m-1} \frac{1}{m} \delta_{2\pi k/m}$  is the uniform distribution on the vertices of a regular  $m$ -gone (with a vertex at position  $(0, 0)$ ), then all the  $b_k$  are null,  $a_0(\mu) = 1/(2\pi)$ ,  $a_k(\mu) = \pi^{-1} \mathbf{1}_{k \in m\mathbb{N}^*}$ .

Of course, to decide whether a given pair  $((a_k, k \geq 0), (b_k, k \geq 1))$  corresponds to a pair  $((a_k(\nu), k \geq 0), (b_k(\nu), k \geq 1))$  for some  $\nu \in \mathcal{M}_{\mathcal{T}}$  is a difficult task: there does not exist in the literature any characterisation of Fourier series of non negative measures. The case of measures having a density with respect to the Lebesgue measure is discussed in Section 5.2.

The area of a convex  $\mathcal{C}_\mu$  has an expression in terms of  $\text{Coeffs}(\mu)$ . This formula can be deduced from Hurwitz [14, p.372-373], where it is given using a parametrisation of the border of the convex. In our settings, writing  $\mathcal{A}(\mu)$  for the area of  $\mathcal{C}_\mu$ , it translates :

$$\mathcal{A}(\mu) = \frac{1}{4\pi} - \frac{\pi}{2} \sum_{k \geq 2} \frac{a_k^2(\mu) + b_k^2(\mu)}{k^2 - 1}. \quad (8)$$

This equation can be proved, as did Hurwitz, starting from Green's theorem stating that :

$$\mathcal{A}(\mu) = \int_0^1 x_\mu(t) \frac{dy_\mu(t)}{dt} dt = - \int_0^1 y_\mu(t) \frac{dx_\mu(t)}{dt} dt, \quad (9)$$

this can be easily seen (at least at an intuitive level), since  $-\int_0^1 y_\mu(t) \frac{dx_\mu(t)}{dt} dt$  represents the area below the curve above the convex minus the area below the curve at the bottom of it, and a similar horizontal decomposition provides the second formula.

In fact, Hurwitz formula is stated in [14] for convexes having smooth border, and in particular, no part of the border can be segments. As a matter of fact, this formula remains valid for every convex in  $\text{Conv}(1)$  (cf. Corollary 3.8). Rewriting (9) using (6) gives

$$\begin{aligned} \mathcal{A}(\mu) &= \int_0^1 \int_0^t \cos(F_\mu^{-1}(u)) du \sin(F_\mu^{-1}(t)) dt \\ &= \int_0^{2\pi} \int_0^y \cos(x) dF_\mu(x) \sin(y) dF_\mu(y) \\ &= \mathbb{E}(\cos(X) \sin(X') 1_{X \leq X'}) . \end{aligned} \quad (10)$$

where  $X$  and  $X'$  are two independent copies of  $X_\mu$ . To see the compatibility between the formulas (10) and (8), let us start from (8), and get (10) by logical inferences. By (6)

$$\mathcal{A}(\mu) = \frac{1}{4\pi} - \frac{1}{2\pi} \sum_{k \geq 2} \frac{\mathbb{E}(\cos(kX))^2 + \mathbb{E}(\sin(kX))^2}{k^2 - 1} \quad (11)$$

$$= \frac{1}{4\pi} - \frac{1}{2\pi} \sum_{k \geq 2} \frac{\mathbb{E}(\cos(k(X - X')))}{k^2 - 1}. \quad (12)$$

Now, notice that for any  $x \in \mathbb{R}$ ,

$$\sum_{k \geq 2} \frac{\cos(kx)}{k^2 - 1} = \frac{\cos(x)}{4} - \frac{(\pi - (x \bmod 2\pi))}{2} \sin(x) + \frac{1}{2}. \quad (13)$$

This leads to

$$\mathcal{A}(\mu) = - \frac{\mathbb{E}(\cos(X - X') - 2 \sin(X - X')(\pi - (X - X' \bmod 2\pi)))}{8\pi}.$$

Using that  $\mathbb{E}(\sin(X_\mu)) = \mathbb{E}(\cos(X_\mu)) = 0$ , one sees that  $\mathbb{E}(\cos(X - X')) = \mathbb{E}(\sin(X - X')) = \mathbb{E}((X - X') \sin(X - X')) = 0$ , and then, since  $X - X' \bmod 2\pi = (X - X') + 2\pi 1_{X \leq X'}$ ,

$$\mathcal{A}(\mu) = \frac{1}{2} \mathbb{E}(\sin(X' - X) 1_{X \leq X'}), \quad (14)$$

which is indeed the same formula as (10). Notice that Hurwitz [13] deduced the isoperimetric inequality from this equation with a proof simply requiring an equivalent of Wirtinger's inequality.

## 2.4 Convergence of discrete convexes and an application to statistics

The following proposition is similar to the convergence of empirical process; it provides a criterion to test if a sample of r.v. follows a given distribution.

Consider  $X_1, \dots, X_n$  i.i.d. having distribution  $\mu$  with support in  $[0, 2\pi]$ . The empirical CDF associated with this sample is defined by  $F_n(x) = n^{-1} \#\{i : X_i \leq x\}$ . The law of large number ensures that  $F_n \rightarrow F_\mu$  pointwise in probability, and  $(n^{1/2}|F_n(x) - F_\mu(x)|, x \in [0, 2\pi])$  converges in distribution in  $D[0, 2\pi]$  to  $(b(F_\mu(x)), x \in [0, 2\pi])$  where  $b$  is a standard Brownian bridge (see Billingsley [5, Theorem 14.3]).

Now assume that the  $X_i$  take their values in  $\mathcal{T}$  (when sorting the variables,  $0 = 2\pi$  is considered to be the smallest), and let  $\hat{X}_1, \dots, \hat{X}_n$  be the sequence  $X_1, \dots, X_n$  in increasing order. Consider the function  $Z_n : [0, 1] \rightarrow \mathbb{C}$  defined by  $Z_n(0) = 0$ ,

$$Z_n(k/n) = \frac{1}{n} \sum_{j=1}^k \exp(i\hat{X}_j), \quad \text{for } k \in \{1, \dots, n\},$$

and extended by linear interpolation between the points  $(k/n, k \in \{0, \dots, n\})$ . Also, let  $C_n := \{Z_n(t), t \in [0, 1]\}$ , be the range of  $Z_n$ : this is the empirical curve associated with the distribution  $\mu$ , when the steps have been sorted as we said. The curve  $C_n$  is not necessarily simple and may contain up to 1 self-intersection;  $C_n$  belongs to  $\text{Conv}(1)$  if and only if  $\sum_{j=1}^n e^{iX_j} = 0$ . For  $\theta \in [0, 2\pi]$ , let  $N_n(\theta) = \#\{i, X_i \leq \theta\}$  the number of variables smaller than  $\theta$ . Notice that

$$\text{Ext}(C_n) = \{Z_n(N_n(\theta)), \theta \in [0, 2\pi]\}. \quad (15)$$

Set, for any  $\theta \in [0, 2\pi]$ ,

$$W_n(\theta) := \sqrt{n} [Z_n(N_n(\theta)) - Z_\mu(F_\mu(\theta))].$$

This process measures the difference between  $Z_n$  and its limit.

Denote by  $\pi_1(z) = \Re(z)$ ,  $\pi_2(z) = \Im(z)$  and  $\pi(z) = (\pi_1(z), \pi_2(z))$ .

**Theorem 2.7.** 1) *The following convergence*

$$\pi(W_n(\theta), \theta \in [0, 2\pi]) \xrightarrow[n]{(d)} (G_\theta, \theta \in [0, 2\pi]) \quad (16)$$

holds in  $(D[0, 1], \mathbb{R}^2)$ , where  $G$  is a centred Gaussian process whose finite dimensional distribution are given in Section 6.1, in Formula 40.

2) *The sequence of rescaled distances  $\sqrt{n}d_H(C_n, C_\mu)$  converges in distribution.*

The following Corollary is a direct consequence of Theorem 2.7, but we think that it deserves to be mentioned separately since it represents in some sense a “law of large numbers” type theorem for our random polygons (see Figure 4).

**Corollary 2.8.** *If  $\mu \in \mathcal{M}_{\mathcal{T}}^0$  then:*

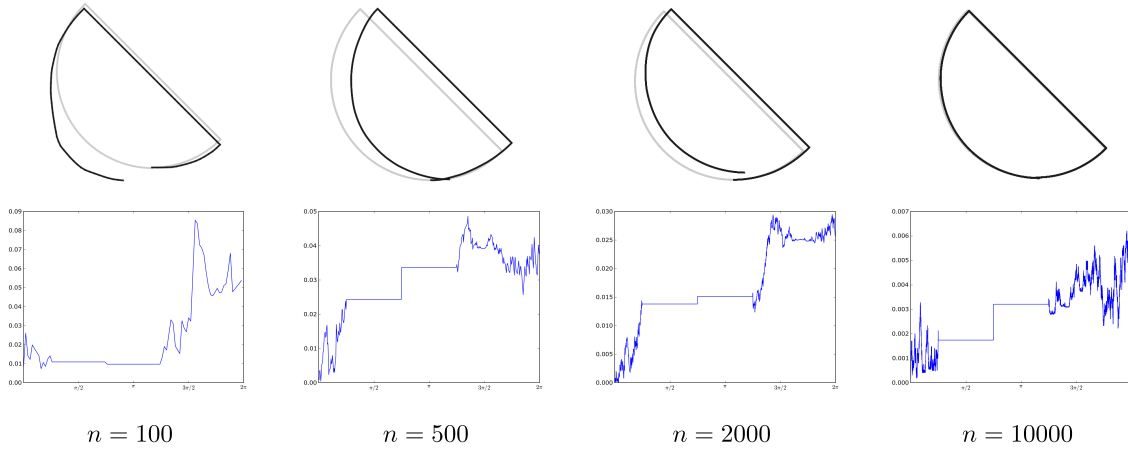


Figure 4: Convergence towards the half-circle. The first row of figures describes the discrete convex of size  $n$  (in black) compared to the limit convex (in gray). The second row displays the distance between the discrete convex and its limit.

(1) *The following convergence holds in distribution in  $D[0, 1]$ :*

$$(Z_n(N_n(\theta)), \theta \in [0, 2\pi]) \xrightarrow[n]{(d)} (Z_\mu(F_\mu(\theta)), \theta \in [0, 2\pi]). \quad (17)$$

(2)  $d_H(C_n, C_\mu) \rightarrow 0$  in probability.

Besides, a direct proof of the Corollary 2.8 which does not use Theorem 2.7 proceeds as follows: first, a convergence of the finite dimensional distributions (FDD) corresponding to the convergence stated follows the law of large numbers. Then, for a  $\varepsilon > 0$ , choosing properly  $k$  and the points  $(\theta_1, \dots, \theta_k)$  such that the union of the segments  $C_\varepsilon := \cup_{i=0..k-1} [Z_\mu(F_\mu(\theta_i)), Z_\mu(F_\mu(\theta_{i+1}))]$  has a length larger than  $1 - \varepsilon$ . From there, (2) follows since for  $n$  large enough,  $|Z_n(N_n(\theta_i)) - Z_\mu(F_\mu(\theta_i))|$  goes to 0 in probability for any  $i \leq k$ . This implies that the union of the segments  $C'_n = \cup_i [Z_n(N_n(\theta_i)), Z_n(N_n(\theta_{i+1}))]$  has a total length larger than  $1 - 2\varepsilon$  for  $n$  large enough, with probability going to 1. Further, for those  $n$ ,  $d_H(C_n, C'_n) \leq 2\varepsilon$  since  $C_n$  has length 1.

The proof of Theorem 2.7 is postponed at the appendix.

### 3 Operations on measures and on convexes

Some natural operations exist on  $\mathcal{M}_{\mathcal{T}}^0$ , for instance the mixture and the convolution. Theorem 2.1 transports these operations naturally on the set of convexes  $\text{Conv}(1)$ . In this section, we provide some consequences of this which seem to be unknown. Hence a mixture is sent by  $\mathcal{C}$  on a Minkowski sum (Proposition 3.2) ; this has some consequences in term of Minkowski symmetrisation (Theorem 3.6). Another interesting fact is that the convolution of measures allows one to define a notion of convolution of convexes (acting somehow on the radius of curvature) which also seems to be new, as well as a notion of symmetrisation by convolution.

### 3.1 Convolutions and mixture of measures and of convexes

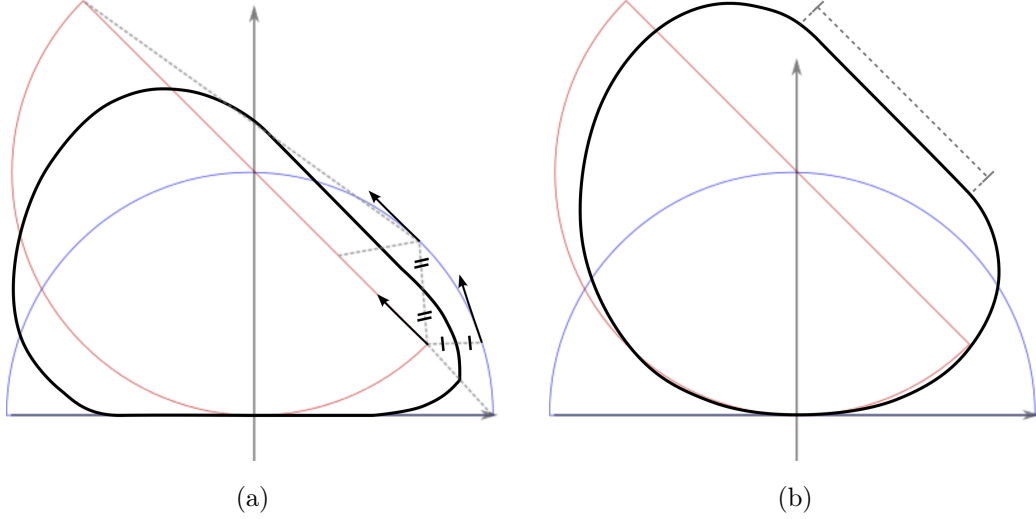


Figure 5: Construction of the (a) mixture and (b) convolution of two half-circles. Notice that every point of the mixed convex is the barycentre of two points of the original half-circles, and that the convex obtained by convolution possesses a linear segment whose angle corresponds to the sum of the angles of the segments in the original half-circles.

We begin this section by a simple observation, that can be proved using the characterisation of measures by their Fourier transform:  $\mathcal{M}_{\mathcal{T}}^0$  is stable by mixture and by convolution :

- (1) The mixture: if  $\mu$  and  $\nu$  belong to  $\mathcal{M}_{\mathcal{T}}^0$  then for any  $\lambda \in [0, 1]$ ,  $\lambda\mu + (1 - \lambda)\nu \in \mathcal{M}_{\mathcal{T}}^0$ .
- (2) The convolution : if  $\mu$  and  $\nu$  belong to  $\mathcal{M}_{\mathcal{T}}^0$  then  $\mu \star_{\mathcal{T}} \nu \in \mathcal{M}_{\mathcal{T}}^0$ , where  $(\star)$  denotes the convolution in  $\mathcal{M}_{\mathcal{T}}$  (or equivalently the law of the sum taken modulo  $2\pi$ ). The same conclusion holds even if only  $\mu \in \mathcal{M}_{\mathcal{T}}^0$ .

**Definition 3.1.** Let  $\mathcal{C}_{\mu}$  and  $\mathcal{C}_{\nu}$  be two convexes in  $\text{Conv}(1)$  and  $\lambda \in (0, 1)$ .

- (1) We call mixture of  $\mathcal{C}_{\mu}$  and of  $\mathcal{C}_{\nu}$  with weights  $(\lambda, 1 - \lambda)$ , the convex  $\mathcal{C}_{\lambda\mu + (1-\lambda)\nu}$ .
- (2) We call convolution of the convexes  $\mathcal{C}_{\mu}$  and  $\mathcal{C}_{\nu}$ , the convex  $\mathcal{C}_{\mu \star_{\mathcal{T}} \nu}$ .

In Section 3.2 we observe that mixtures of convexes and Minkowski sums is the same operation, and make some deductions. Section 3.3 concerns the convolution of measures, and an associated notion of convolution of convexes; as far as we know, this operation is new, and allows one to define a new notion of symmetrisation of convexes.

### 3.2 Mixtures of convexes / Minkowski sum

Let  $A$  and  $B$  be two subsets of  $\mathbb{R}^2$ . The Minkowski sum of  $A$  and  $B$  is the set  $A + B = \{a + b : a \in A, b \in B\}$ . Further, for any  $\lambda$ , write  $\lambda A = \{\lambda a : a \in A\}$ . In this section, it is necessary to

manipulate convex sets as well as convex borders. In the following, let  $B_\mu$  be the convex hull of  $\mathcal{C}_\mu$ . We have:

**Proposition 3.2.** *Let  $\nu, \mu \in \mathcal{M}_T^0$ ,  $\lambda \in [0, 1]$ . We have*

$$B_{\lambda\mu+(1-\lambda)\nu} = \lambda B_\mu + (1-\lambda)B_\nu$$

*which means that the mixture of convexes and the Minkowski sum are the same, and that the convex associated to a mixture, is the associated mixture of convexes.*

This proposition (see Figure 5) implies that the borders verify :

$$\mathcal{C}_{\lambda\mu+(1-\lambda)\nu} = \partial(\text{convex hull}(\lambda\mathcal{C}_\mu + (1-\lambda)\mathcal{C}_\nu))$$

The mixture simply operates on probability distributions, or on the CDF, but not on generalised inverse distribution function that encodes more simply the convexes as one can see in (1).

**Proof.** Recall the characterisation given in Lemma 2.3. Write

$$\begin{aligned} Z_{\lambda\mu+(1-\lambda)\nu}(F_{\lambda\mu+(1-\lambda)\nu}(t)) &= \lambda \int_0^t \exp(it) d\mu(t) + (1-\lambda) \int_0^t \exp(it) d\nu(t) \\ &= \lambda Z_\mu(F_\mu(t)) + (1-\lambda) Z_\nu(F_\nu(t)). \end{aligned} \quad (18)$$

The extremal points of  $\mathcal{C}_{\lambda\mu+(1-\lambda)\nu}$  are then obtained as particular barycentres of extremal points of  $\mathcal{C}_\mu$  and  $\mathcal{C}_\nu$ . When both  $\mu$  and  $\nu$  have a density, this says that the point in  $\mathcal{C}_{\lambda\mu+(1-\lambda)\nu}$  where the tangent has direction  $\theta$  is obtained as the barycentre of corresponding points in  $\mathcal{C}_\mu$  and  $\mathcal{C}_\nu$  where the tangent has the same direction. This implies that  $B_{\lambda\mu+(1-\lambda)\nu} \subset \lambda B_\mu + (1-\lambda)B_\nu$ .

Let us establish the other inclusion by using the fact that the convexes are characterised by their supporting half-planes (Lemma 2.4). For every  $t \in [0, 2\pi]$ , let  $D_\mu(t)$  be the line passing through  $Z_\mu(F_\mu(t))$  making an angle  $t$  relatively to the  $x$ -axis. The line  $D_\mu(t)$  defines a supporting half-plane  $H_\mu(t)$  for  $\mathcal{C}_\mu$ . Moreover, since  $\mathcal{C}_\mu$  is convex, this half-plane is minimal for the inclusion with regard to that property (of making an angle  $t$  relatively to the  $x$ -axis). Considering that the points in (18) all belong to their associated half-plane, these half-planes verify the following equation for the Minkowski addition :

$$H_{\lambda\mu+(1-\lambda)\nu}(t) = \lambda H_\mu(t) + (1-\lambda)H_\nu(t).$$

Now, the left-hand side represents a supporting half-plane for  $\mathcal{C}_{\lambda\mu+(1-\lambda)\nu}$  and the right-hand side another supporting half-plane for  $\lambda\mathcal{C}_\mu + (1-\lambda)\mathcal{C}_\nu$ . According to Lemma 2.4 the convex sets they enclose are equal.  $\square$

Hence the convex  $\mathcal{C}_{\lambda\mu+(1-\lambda)\nu}$  is a convex having perimeter 1 as all convexes of  $\text{Conv}(1)$ . This implies for free that the perimeter of the Minkowski sum  $\lambda\mathcal{C}_\mu + (1-\lambda)\mathcal{C}_\nu$  is 1.

**Theorem 3.3.** Let  $\mu$  and  $\nu$  in  $\mathcal{M}_T^0$ . For any  $\lambda \in [0, 1]$ , we have the Brunn-Minkowski inequality

$$\mathcal{A}(\lambda\mu + (1-\lambda)\nu)^{1/2} \geq \lambda\mathcal{A}(\mu)^{1/2} + (1-\lambda)\mathcal{A}(\nu)^{1/2}$$

(which ensures that  $\mathcal{A}(\lambda\mu + (1-\lambda)\nu) \geq \min\{\mathcal{A}(\mu), \mathcal{A}(\nu)\}$ ).

**Proof.** Let  $\eta = \lambda\mu + (1-\lambda)\nu$ . For any measure  $\mu$  denote by

$$R_\mu := \sum_{k \geq 2} \frac{a_k^2(\mu) + b_k^2(\mu)}{k^2 - 1}$$

the value such that  $\mathcal{A}(\mu) = \frac{1}{4\pi} - \frac{\pi}{2}R_\mu$ . We have

$$a_n(\eta) = \lambda a_n(\mu) + (1-\lambda)a_n(\nu)$$

$$b_n(\eta) = \lambda b_n(\mu) + (1-\lambda)b_n(\nu).$$

Hence,

$$a_n^2(\eta) + b_n^2(\eta) = \lambda^2(a_n^2(\mu) + b_n^2(\mu)) + (1-\lambda)^2(a_n^2(\nu) + b_n^2(\nu)) \quad (19)$$

$$+ \lambda(1-\lambda)(2a_n(\mu)a_n(\nu) + 2b_n(\mu)b_n(\nu)). \quad (20)$$

Now, divide by  $n^2 - 1$  on both sides, and take the sum over  $n \geq 2$ , and obtain that

$$R_\eta = \lambda^2 R_\mu + (1-\lambda)^2 R_\nu + 2\lambda(1-\lambda)Q.$$

with  $Q = \sum_{n \geq 2} (a_n(\mu)a_n(\nu) + b_n(\mu)b_n(\nu)) / (n^2 - 1)$ . Now

$$Q \leq \sqrt{R_\mu R_\nu}. \quad (21)$$

Indeed:

$$\langle ((a_k, b_k), k \geq 2), ((a'_k, b'_k), k \geq 2) \rangle = \sum_{k \geq 2} \frac{a_k a'_k + b_k b'_k}{k^2 - 1}.$$

is a scalar product on the set  $(\mathbb{R}^2)^\mathbb{N}$  of sequences of pairs of real numbers  $((a_k, b_k)_{k \geq 2})$ . Denoting  $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$  the associated norm, one sees that  $\|((a_k(\mu), b_k(\mu)), k \geq 2)\|^2 = R_\mu$ , further (21) appears to be a consequence of the Cauchy-Schwarz inequality. Finally, we get

$$R_\eta \leq \lambda^2 R_\mu + (1-\lambda)^2 R_\nu + 2\lambda(1-\lambda)\sqrt{R_\nu R_\mu},$$

follows that

$$\mathcal{A}(\eta) \geq \lambda^2 \mathcal{A}(\mu) + (1-\lambda)^2 \mathcal{A}(\nu) + 2\lambda(1-\lambda) \left( \frac{1}{4\pi} - \frac{\pi}{2} \sqrt{R_\nu R_\mu} \right),$$

to end the proof, check (by taking the square and expanding, for example) that

$$\frac{1}{4\pi} - \frac{\pi}{2} \sqrt{R_\nu R_\mu} \geq \sqrt{\frac{1}{4\pi} - \frac{\pi}{2} R_\nu} \sqrt{\frac{1}{4\pi} - \frac{\pi}{2} R_\mu} = \sqrt{\mathcal{A}(\mu) \mathcal{A}(\nu)}$$

The second assertion is a consequence of the concavity of  $\lambda \mapsto \lambda \mathcal{A}(\mu)^{1/2} + (1-\lambda) \mathcal{A}(\nu)^{1/2}$ .  $\square$



### 3.2.1 Minkowski symmetrisation and measure symmetrisation

Let  $K$  be a convex of  $\mathbb{R}^2$  and  $u \in \mathbb{R}^2$ ,  $|u| = 1$ . We denote by  $\pi_u \in O(2)$  the reflection with respect to the straight line passing through the origin and orthogonal to  $u$ , i.e.  $\pi_u(x) = x - 2\langle x, u \rangle u$ . The *Minkowski* (or Blaschke) *symmetrisation* of  $K$  is the convex  $S_u(K) = \frac{1}{2}(\pi_u K + K)$ . The same operation can be defined over  $\mathbb{C}$  instead: for  $u = e^{i\theta}$ , the Minkowski (or Blaschke) symmetrisation of  $K$  with respect to  $\theta$  (or  $e^{i\theta}$ ) is the map  $(K, \theta) \mapsto \frac{e^{i\theta}}{2}(\overline{e^{-i\theta}K} + e^{-i\theta}K)$ , where  $\bar{z}$  is the complex conjugate to  $z$ .

Now, let  $\theta \in [0, 2\pi]$ ,  $\mu \in \mathcal{M}_{\mathcal{T}}^0$ , and set  $\mu_\theta$  be the distribution of  $X_\mu + \theta \pmod{2\pi}$ . Since  $\mathbb{E}(\exp(i(X_\mu + \theta))) = e^{i\theta} \mathbb{E}(\exp(iX_\mu))$ ,  $\mu_\theta$  is in  $\mathcal{M}_{\mathcal{T}}^0$ . The convex  $\mathcal{C}_{\mu_\theta}$  can be obtained from  $\mathcal{C}_\mu$  by a rotation (of angle  $-\theta$ ) composed by a translation.

For any  $\nu \in \mathcal{M}_{\mathcal{T}}^0$ , set  $\overleftarrow{\nu} = \nu(2\pi - \cdot)$ .

**Definition 3.4.** *The symmetrisation of  $\nu$  with respect to  $\theta$  is the map that with a measure  $\nu$  associates*

$$S(\nu(\theta)) = \frac{1}{2}(\nu(\theta) + \overleftarrow{\nu(\theta)}). \quad (22)$$

*This definition extends to convexes: the symmetrisation by mixture of  $\mathcal{C}_\nu$  with respect to the direction  $e^{i\theta}$  is defined to be  $\mathcal{C}_{S(\nu(\theta))}$ .*

A direct consequence of Proposition 3.2 is the following:

**Proposition 3.5.** *The symmetrisation by mixture with respect to the direction  $e^{i\theta}$  coincides with the Minkowski symmetrisation with respect to  $u = e^{i\theta}$ .*

Again Theorem 2.7 allows one to have a new point of view on this symmetrisation.

We introduce a notation for the measure (and the convex) obtained by performing some successive rotations  $\theta_1, \dots, \theta_k$  alternatively followed by symmetrisation ( $\nu_0 = \nu$ ,  $\nu_1 = S(\nu_0(\theta_1))$ ,  $\nu_k = S(\nu_{k-1}(\theta_k))$ ) where of course, the considered  $\theta_k$  will be clear from the context.

**Theorem 3.6.** *For any  $\theta \in [0, 2\pi]$ , any  $\nu \in \mathcal{M}_{\mathcal{T}}^0$ ,*

- (1) *the convex  $\mathcal{C}_{S(\nu(\theta))}$  has the same perimeter as  $\mathcal{C}_\nu$  (that is 1),*
- (2) *the area does not decrease:  $\mathcal{A}(S(\nu(\theta))) \geq \mathcal{A}(\nu)$*
- (3) *for any  $k \geq 0$ , there exists  $\theta_1, \dots, \theta_k \in [0, 2\pi]$  such that  $d_H(\mathcal{C}_{\nu_k}, \text{Circle}(i/(2\pi), 1/(2\pi))) \leq 2^{-k}\pi$ . item[(4)] Among all convexes with perimeter 1, the circle has the largest area.*

Properties (1), (2), (4) are quite classical; we provide direct probabilistic proof below. Statement (3) which gives a bound on the speed of convergence to the ball for well chosen symmetrisation, is known and true in  $\mathbb{R}^n$  (see Klartag [15, Theorem 1.3]). The proof we provide here in  $\mathbb{R}^2$  is much simpler than Klartag's one (but Klartag's result holds in  $\mathbb{R}^n$ , for any  $n \geq 2$ ).

**Proof.** First, (4) is clearly a consequence of the three first points. Our proof of (2) uses Theorem 3.3 used the Hurwitz formula, which implies clearly the isoperimetric inequality directly.

Now, since  $S(\nu(\theta)) \in \mathcal{M}_T^0$ , and then  $\mathcal{C}_{S(\nu(\theta))} \in \text{Conv}(1)$ . This proves (1). Now, Theorem 3.3 implies (2) since  $\mathcal{A}(\nu) = \mathcal{A}(\nu(\theta)) = \mathcal{A}(\overleftarrow{\nu(\theta)})$ . Let us prove (3). For this let  $L = [X_1, \dots, X_l]$  for some  $l \geq 1$ , a list of r.v. with distribution  $\nu_1, \dots, \nu_l$ . We say that  $\nu$  is the equi-mixture of  $L$  if  $\nu = \frac{1}{l}(\nu_1 + \dots + \nu_l)$ .

Take  $X \sim \nu$ .  $S(\nu(\theta))$  is the equi-mixture of  $[X + \theta \bmod 2\pi, -X - \theta \bmod 2\pi]$ . Therefore using that  $(a \bmod 2\pi) + b \bmod 2\pi = (a + b) \bmod 2\pi$ ,  $S_{\nu_2}$  is the equi-mixture of  $[X + \theta_1 \pm \theta_2 \bmod 2\pi, -X - \theta_1 \pm \theta_2 \bmod 2\pi]$ . Iterating this, one finds that  $S_{\nu_k}$  is the equi-mixture of  $[X + \theta_1 \pm \theta_2 \pm \dots \pm \theta_k \bmod 2\pi, -X - \theta_1 \pm \theta_2 \pm \dots \pm \theta_k \bmod 2\pi]$ . If  $\theta_k = (2\pi)/2^{k-1}$  (we take  $\theta_1 = 2\pi$ ) then  $S_{\nu_k}$  is the equi-mixture of  $\mu_1$  and  $\mu_2$ , where  $\mu_1$  and  $\mu_2$  are the respective equi-mixture of  $[X + \theta_1 \pm \theta_2 \pm \dots \pm \theta_k \bmod 2\pi]$  and of  $[-X - \theta_1 \pm \theta_2 \pm \dots \pm \theta_k \bmod 2\pi]$ . Now, both  $\mu_1$  and  $\mu_2$  converges to  $\text{uniform}[0, 2\pi]$ : For this, consider the set of interval  $(I_n = [2\pi n 2^{-k-1}, 2\pi(n+1)2^{-k-1}, 0 \leq n \leq 2^{k-1}-1])$ . It is easy to see that, for  $j \in \{1, 2\}$ ,  $\mu_j(I_n) = 2^{k-1}$  for any  $n$ . Hence  $F_{\mu_1}(2\pi n 2^{-k+1}) = n 2^{-k+1}$  for any  $n$ . Therefore, since  $F_{\mu_1}$  is increasing, we have that  $\|F_{\mu_j} - F\|_\infty \leq 2^{-k+1}$ , for  $F_v(x) = x/(2\pi)$ , the CDF of  $\text{uniform}[0, 2\pi]$ , which gives at once  $\|F_{\nu_k} - F_v\|_\infty \leq 2^{-k+1}$ .

In this case, in view of  $F$ , the right inverses  $F_{\nu_k}^{-1}$  and  $F_v^{-1}$  are close:

$$\|F_{\nu_k}^{-1} - F_v^{-1}\|_\infty \leq 2^{-k+1} 2\pi$$

Thanks to (1),

$$\begin{aligned} |Z_{\nu_k}(t) - Z_v(t)| &\leq \int_0^t |\exp(iF_{\nu_k}^{-1}(u)) - \exp(iF_v^{-1}(u))| du \\ &\leq \int_0^t |F_{\nu_k}^{-1}(u) - F_v^{-1}(u)| du \end{aligned}$$

and therefore  $\|Z_{\nu_k}(t) - Z_v(t)\|_\infty \leq 2^{-k}\pi$ .  $\square$

### 3.3 Convolution of measures / Convolution of convexes

In fact,  $\mathcal{C}_{\mu \star_\tau \nu}$  is obtained as a kind of convolution of  $\mathcal{C}_\mu$  and  $\mathcal{C}_\nu$ . As seen earlier, if  $\mu$  has a density  $f_\mu$ , then  $f_\mu(\theta)$  represents the radius of curvature of  $\mathcal{C}_\mu$  at the position  $F_\mu(\theta)$ . Therefore, the radius of curvature  $R_\theta$  of  $\mathcal{C}_{\mu \star_\tau \nu}$  at position  $F_{\mu \star_\tau \nu}(\theta)$  is the convolution of the radii of curvature of  $\mathcal{C}_\mu$  and  $\mathcal{C}_\nu$ :

$$R_\theta = \int_0^{2\pi} f_\mu(x) f_\nu((\theta - x) \bmod 2\pi) dx.$$

**Theorem 3.7.** *Let  $\mu$  and  $\nu$  in  $\mathcal{M}_T^0$ . The convolution does not decrease the area*

$$\mathcal{A}\left(\mu \star_\tau \nu\right) \geq \max\{\mathcal{A}(\mu), \mathcal{A}(\nu)\}.$$

*Since  $u = \text{uniform}[0, 2\pi]$  is an absorbing point for  $\star_\tau$ , and  $\mathcal{C}_u$  is the circle of perimeter 1, this implies the isoperimetric inequality :  $\mathcal{A}(u) \geq \mathcal{A}(\nu)$ ,  $\forall \nu \in \mathcal{M}_T^0$ .*

**Proof.** Consider  $X$  and  $Y$  two independent r.v. on the same probability space, such that  $X \sim \mu$ ,  $Y \sim \nu$ . Let  $\eta = \mu \star_T \nu$ . By a simple expansion of  $\cos(n(X + Y))$  and of  $\sin(n(X + Y))$  we get

$$\begin{aligned} a_n(\eta) &= a_n(\mu)a_n(\nu) - b_n(\mu)b_n(\nu) \\ b_n(\eta) &= b_n(\mu)a_n(\nu) + a_n(\mu)b_n(\nu). \end{aligned}$$

Since  $\cos(kX)$  and  $\sin(kX)$  have a non-negative variance,

$$a_n^2(\mu) + b_n^2(\mu) = \mathbb{E}(\cos(nX))^2 + \mathbb{E}(\sin(nX))^2 \leq \mathbb{E}(\cos^2(nX) + \sin^2(nX)) = 1.$$

Hence,

$$\begin{aligned} a_n^2(\eta) + b_n^2(\eta) &= (a_n^2(\mu) + b_n^2(\mu))(a_n^2(\nu) + b_n^2(\nu)) \\ &\leq \min\{a_n^2(\mu) + b_n^2(\mu), a_n^2(\nu) + b_n^2(\nu)\}, \end{aligned}$$

from which we deduce  $R_\eta \leq \min\{R_\mu, R_\nu\}$ . Again, the conclusion follows from (8).  $\square$

**Corollary 3.8.** *Let  $\mu \in \mathcal{M}_T^0$ . Then the formula (8) for  $\mathcal{A}(\mu)$  holds.*

**Proof.** Formula (8) is valid when  $\mu$  admits a  $\mathcal{C}^1$  density. We here just assume that  $\mathbb{E}(e^{iX_\mu}) = 0$ . Let  $N$  be a Gaussian centred r.v. with variance 1, and let  $N_k = N/\sqrt{k} \mod 2\pi$  for  $k \geq 1$ , and  $\mu_k = \mu * N_k$ . Clearly  $\mu_k \in \mathcal{M}_T^0$ , and  $\mu_k \xrightarrow[n]{(weak)} \mu$  which implies  $\mathcal{A}(\mu_k) \rightarrow \mathcal{A}(\mu)$ . Now,

$$\forall n \in \mathbb{Z}, \quad \mathbb{E}(e^{inN_k}) = \mathbb{E}(e^{in(N/\sqrt{k} \mod 2\pi)}) = \mathbb{E}(e^{inN/\sqrt{k}}) = e^{-\frac{n^2}{2k}}.$$

Then the Fourier coefficients of  $N_k$  verify then  $a_n = e^{-\frac{n^2}{2k}}$  and  $b_n = 0$ . Since  $\mu_k$  admits a  $\mathcal{C}^\infty$  density function, and as a corollary of the proof of Theorem 3.7 :

$$\mathcal{A}(\mu_k) = \frac{1}{4\pi} - \frac{\pi}{2} \sum_{n \geq 2} \frac{(a_n^2(\mu) + b_n^2(\mu)) e^{-\frac{1}{2k}n^2}}{n^2 - 1}$$

As a consequence of the dominated convergence theorem,  $\mathcal{A}(\mu_k)$  converges to the right hand side of (8), which corresponds to the result.  $\square$

**Definition 3.9.** *A measure  $\nu$  in  $\mathcal{M}_T$  is said to be  $c$ -stable (for some  $c > 0$ ) if for  $X_\nu$  and  $X'_\nu$  two independent r.v. under  $\nu$ ,*

$$X_\nu + X'_\nu \mod 2\pi \stackrel{(d)}{=} cX_\nu \mod 2\pi. \quad (23)$$

This qualification of “stable” comes from the standard notion of probability theory where the same question is studied without the additional operation  $\mod 2\pi$  (see Feller [11, Section VI], and notice that  $c$  is not the index of a stable distributions in  $\mathbb{R}$ ).

The following Proposition due to Lévy [17, p.11] identifies the set of 1-stable distributions.

**Proposition 3.10.** *The only 1-stable measures is uniform $[0, 2\pi]$ , the Dirac measure at 0, and the family, indexed by  $m \geq 1$ , of uniform measures on  $\{k2\pi/m, k = 0, \dots, m-1\}$ .*

As far as we know, for  $c \neq 1$ ,  $c$ -stable distributions – which are interesting by themselves – do not have an interest in terms of convexes.

We say that a distribution  $\nu$  is in the  $2\pi$ -domain of attraction of a distribution  $\mu$  (and note  $\nu \in \text{DA}(\mu)$ ) if for a family  $(X_i, i \geq 1)$  of i.i.d. r.v. under  $\nu$ , there exists  $\theta \in [0, 2\pi]$  such that

$$\sum_{i=1}^n (X_i - \theta) \mod 2\pi \xrightarrow[n]{(d)} X_\mu.$$

Denote by  $\text{DA}$  the set of measures  $\mu$  limit of such sequences.

**Proposition 3.11.** (1) *The set  $\text{DA}$  coincides with the set of 1-stable distributions.*

(2) *For any  $\nu \in \mathcal{M}_{\mathcal{T}}^0$ , there exists  $\theta \in [0, 2\pi]$  and a unique  $\mu$  1-stable such that  $\nu \in \text{DA}(\mu)$ .*

**Proof.** (1) If  $\nu$  is a 1-stable distribution, and if  $(X_i, i \geq 1)$  are i.i.d. and taken under  $\nu$ , then it is easily seen that  $X_1 + \dots + X_n \mod 2\pi \stackrel{(d)}{=} X_1$ . Therefore, all 1-stable distributions are in  $\text{DA}$ .

Conversely, assume that  $(X_i, i \geq 1)$  are i.i.d., distributed according to  $\nu$ , and that  $\sum_{i=1}^n (X_i - \theta) \mod 2\pi \xrightarrow[n]{(d)} \mu$ . Cutting the sum on the left-hand side in two parts, we see that  $\mu$  is a solution of  $\mu = \mu \star_{\mathcal{T}} \mu$ , that is  $\mu$  is 1-stable.

(2) Take  $(X_i, i \geq 1)$  i.i.d. r.v. under  $\nu$ ,  $\theta \in [0, 2\pi]$ , and compute the limit of the Fourier coefficients of  $\sum_{j=1}^n (X_j - \theta)$  (see also Wilms [27, Thm. 2.1 and Thm. 2.4]).  $\square$

### 3.4 Symmetrisation of convexes by convolution

Let  $\nu \in \mathcal{M}_{\mathcal{T}}^0$  and  $\overleftarrow{\nu} = \nu(2\pi - \cdot)$ . The distribution

$$S_C(\nu) := \nu \star_{\mathcal{T}} \overleftarrow{\nu} \tag{24}$$

is symmetric since  $X_\nu \stackrel{(d)}{=} -X_\nu$ . We call it the *symmetrisation by convolution* of  $\nu$ .

Notice that in the definition of the symmetrisation, replacing  $2\pi$  by some other  $\theta$  (in  $\overleftarrow{\nu}$ ) affects  $S_C(\nu)$  by a simple rotation in  $\mathcal{T}$ .

Denote by  $\nu_1 = S_C(\nu)$ ,  $\nu_2 = S_C(\nu_1)$ , ... Let  $X_n$  be a variable under  $\nu_n$ .

**Proposition 3.12.** *Let  $\nu \in \mathcal{M}_{\mathcal{T}}^0$ , and let  $\mu$  be the unique measure such that  $S_C(\nu)$  belongs to  $\text{DA}(\mu)$ . For  $\theta = \pi$  or  $\theta = 0$  we have*

$$X_n - n\theta \mod 2\pi \xrightarrow[n]{(d)} \mu.$$

**Proof.** First,  $\nu_n$  is the distribution of  $\sum_{i=1}^n X_i - X'_i \mod 2\pi$  for some i.i.d. copies  $X'_i$ s and  $X_i$ 's of  $X_\nu$ . The Fourier coefficients of  $\nu_n$  can then be computed, and they converge to those of a 1-stable distribution as in Proposition 3.11, for  $\theta \in \{0, \pi\}$  since  $X_i - X'_i$  is symmetric.  $\square$

## 4 Extensions

In this section are discussed two natural extensions of the previous considerations. In the short Section 4.1 is evoked the case of compact convexes with any perimeter. In Section 4.2 is investigated the convergence of a trajectory made by i.i.d. increments (with values in  $\mathbb{C}$ ) sorted according to their arguments. At the limit, a centred distribution  $\nu$  on  $\mathbb{C}$  is sent on a convex  $\mathcal{C}_{K(\nu)}$  (for an operator  $K$  defined below).

### 4.1 Convexes with non fixed perimeter

The length of the convex in the construction we gave is 1 because the total mass of all measures in  $\mathcal{M}_{\mathcal{T}}^0$  is 1. Denote by  $\overline{\mathcal{M}_{\mathcal{T}}}^0$  the set of positive measures  $\nu$  with support  $\mathcal{T}$  and such that  $\nu(\mathcal{T}) < +\infty$ . Formula (1), which defines the convex associated with a probability measure extends to these measures, and the convex perimeter  $\text{Peri}(\nu) = \nu(\mathcal{T})$ . A lot of statements given before extend naturally to  $\overline{\mathcal{M}_{\mathcal{T}}}^0$ . Most notably

**Proposition 4.1.** *For any measures  $\nu_1, \nu_2 \in \overline{\mathcal{M}_{\mathcal{T}}}^0$ , any positive numbers  $\lambda_1, \lambda_2$  we have:*

$$\begin{aligned} \text{Peri} \left( \sum_{i=1}^n \lambda_i \nu_i \right) &= \sum_{i=1}^n \lambda_i \text{Peri}(\nu_i) \\ \text{Peri}(\nu_1 \star \nu_2) &= \text{Peri}(\nu_1) \text{Peri}(\nu_2). \end{aligned}$$

*The area of  $\mathcal{C}_{\sum_{i=1}^n \lambda_i \nu_i}$  and of  $\mathcal{C}_{\nu_1 \star \nu_2}$  are still given with the Fourier coefficient of the measures  $\sum_{i=1}^n \lambda_i \nu_i$  and  $\nu_1 \star \nu_2$ , as explained in the proof of Theorem 3.3.*

As we said before, the statement corresponding of measure mixtures, seen as an operation on Minkowski sums, was known.

### 4.2 Measures on $\mathbb{C}$

Theorem 2.1 (Gauss-Minkowski) which sends measures on convexes can be seen thanks to Corollary 2.8 as the convergence of polygonal lines corresponding to some reordered random segments. This sorting according to the argument can be done even if the lengths are not constant, but the mean 0 condition ( $\mathbb{E}(e^{iX_\mu}) = 0$ ) is needed to get a closed convex curve at the limit by the law of large numbers. Let's generalise. Let  $\mu$  be a distribution with support included in  $\mathbb{C}$  with mean 0, but different from  $\delta_0$ . Take a sequence  $W := (W_1, \dots, W_n)$  of i.i.d. r.v. with common distribution  $\mu$ , and let  $\hat{W} := (\hat{W}_1, \dots, \hat{W}_n)$  the list  $W$  sorted according to the arguments of the  $W_i$ 's (if several of them have the same argument but different modulus, then take a uniform random order among them). Let  $S := (S(k), k = 0, \dots, n)$  be the sequence of partial sums

$$S(k) := \sum_{j=1}^k \hat{W}_j, \tag{25}$$

piecewise linearly interpolated between integer points, and let  $\mathbf{C}_n = \{S(nt), t \in [0, n]\}$  be the polygonal line corresponding to the graph of  $S$ .

The distribution  $\mu$  induces a law  $\mathbb{P}_{|W|, \arg(W)}$  for the pair  $(|W|, \arg(W))$ , a law  $\mathbb{P}_{\arg(W)}$  for  $\arg(W)$ ; let  $\mathbb{P}_{|W|, x}$  be a version of the distribution of the modulus of  $|W|$  knowing  $\arg(W) = x$  (this is defined up to a null set under  $\mathbb{P}_{\arg(W)}$ ; for the sake of completeness,  $\mathbb{P}_{|W|, x}$  can be taken to be  $\delta_0$  on the complementary set). We denote by  $m_x$  the mean of  $|W|$  under  $\mathbb{P}_{|W|, x}$ .

Let  $\nu$  be the measure having density  $m/\mathbb{E}(|W|)$  with respect to  $\mathbb{P}_{\arg(W)}$ , that is

$$d\nu(x) = \frac{m_x}{\mathbb{E}(|W|)} d\mathbb{P}_{\arg(W)}(x). \quad (26)$$

The map which sends  $\mu$  onto  $\nu$  will be denoted  $K$ :

$$K(\mu) = \nu. \quad (27)$$

Denote by  $F^{\arg}$  the CDF of  $\arg(W)$ , and by  $F_\nu$  that of the measure  $\nu$ . To end, let  $W_\theta$  denote a r.v.  $W$  under the condition  $\{\arg(W) \leq \theta\}$ .

We here prefer to present a theorem stating the aforementioned convergence; we think that it provides an agreeable way to see the phenomenons into play here.

**Theorem 4.2.** *Consider the model described in the present section. Assume that  $\mu$  is centred ( $\neq \delta_0$ ), and admits a finite covariance matrix and let  $\nu = K(\mu)$ . We have*

$$1) d_H(\mathbf{C}_n/(n\mathbb{E}(|W|)), \mathcal{C}_\nu) \xrightarrow[n]{(d)} 0.$$

2) For any  $\theta$ ,

$$\frac{S(N_n(\theta))}{n\mathbb{E}(|W|)} \xrightarrow[n]{(proba.)} \int_0^\theta e^{it} d\nu(t) = Z_\nu(F_\nu(\theta)). \quad (28)$$

**Remark 4.3.** (a) Prosaically, the previous Theorem says that if  $\mu$  is a centred distribution on  $\mathbb{C}$  having some moments of order 2, the convex associated with  $\mu$  is  $\mathcal{C}_{K(\mu)}$ . This should be true for  $\mu$  having only moments of order 1, but the proof we propose works only when moments of order 2 exist.

(b) According to (26) and Theorem 4.2,  $\mathcal{C}_{K(\nu)}$  is the circle (with radius  $1/(2\pi)$ ) if and only if  $\theta \rightarrow m_\nu(\theta)$  is constant.

(c) The ellipse of equation  $x^2/c^2 + y^2 = R^2$  (with perimeter  $2\pi Rc = 1$ , so that  $Rc = 1/(2\pi)$ ), is obtained in the case where

$$m_\nu(\theta) = \frac{1}{2\pi} \frac{c}{\cos(\theta)^2 + c^2 \sin(\theta)^2}.$$

**Proof.** We first show (2). The cardinality of  $N_n(\theta) := \{1 \leq i \leq n : \arg(W_i) \leq \theta\}$  has the binomial  $(n, F^{\arg}(\theta))$  distribution. It satisfies for any  $\theta$

$$N_n(\theta)/n \xrightarrow[n]{(proba.)} F_\nu(\theta). \quad (29)$$

Conditionally on  $N_n(\theta) = m$ , the (multi)set  $\{\hat{W}_1, \dots, \hat{W}_m\}$  is distributed as a set of  $m$  i.i.d. copies of  $W_\theta$ . Therefore, by the law of large number,

$$\frac{S(N_n(\theta))}{n\mathbb{E}(|W|)} \xrightarrow{\text{proba}} \frac{F^{\text{arg}}(\theta)\mathbb{E}(W_\theta)}{\mathbb{E}(|W|)} = \frac{\mathbb{E}(W1_{\text{arg}(W) \leq \theta})}{\mathbb{E}(|W|)} \quad (30)$$

$$= \frac{\mathbb{E}(|W|e^{i \text{arg}(W)} 1_{\text{arg}(W) \leq \theta})}{\mathbb{E}(|W|)} \quad (31)$$

$$= \int_0^\theta e^{it} \frac{m_t}{\mathbb{E}(|W|)} d\mathbb{P}_{\text{arg}(W)}(t) = Z_\nu(F_\nu(\theta)). \quad (32)$$

A same proof shows that  $L_n(\theta)$ , the length of the curve composed by segments between the positions  $(S(i), 0 \leq i \leq N_n(\theta))$  is, at the first order equal to  $n\mathbb{E}(|W|1_{\text{arg}(W) \leq \theta})/(n\mathbb{E}(|W|))$ . For all  $\epsilon > 0$ , there exists  $\theta_1 < \dots < \theta_k$  such that the convex hull of the  $Z_\nu(F_\nu(\theta_i))$  is at distance at most  $\epsilon$  of  $\mathcal{C}_\nu$ . Now,

$$\mathbb{P} \left( \sup_j \left| \frac{S(N_n(\theta_j))}{n\mathbb{E}(|W|)} - Z_\nu(F_\nu(\theta_j)) \right| > \epsilon \right) \xrightarrow{n \rightarrow \infty} 0.$$

It remains to see that the Hausdorff distance between  $\mathbf{C}_n/(n\mathbb{E}(|W|))$  and the convex hull of the  $\frac{S(N_n(\theta_j))}{n\mathbb{E}(|W|)}$ 's goes to zero in probability. Here it suffices to compute the length  $L_n(\theta_i) - L_n(\theta_{i-1})$  of the curve between  $\theta_{i-1}$  and  $\theta_i$  and show that it converges to  $\mathbb{E}(|W|1_{\theta_{i-1} \leq \text{arg}(W) \leq \theta_i})/(n\mathbb{E}(|W|))$ , namely the length of  $\mathcal{C}_\nu$  between the same angles. But this is a simple application of the law of large numbers.  $\square$

We now discuss of convolution and mixture of convexes.

**Proposition 4.4.** *Let  $X$  and  $Y$  are independent r.v. defined on the same probability space with mean 0 (but non equal to 0 a.s.), and  $\lambda \in [0, 1]$ . Let  $\mu_X$ ,  $\mu_Y$ ,  $\mu_{X.Y}$  and  $\mu_{X+Y}$  be the law of  $X$ ,  $Y$ ,  $X.Y$ ,  $X + Y$ . We have*

$$\mathcal{C}_{\mathbf{K}(\mu_{X.Y})} = \mathcal{C}_{\mathbf{K}(\mu_X)} \star \mathcal{C}_{\mathbf{K}(\mu_Y)} \quad \text{and} \quad \mathcal{C}_{\mathbf{K}(\lambda\mu_X + (1-\lambda)\mu_Y)} = \lambda\mathcal{C}_{\mathbf{K}(\mu_X)} + (1-\lambda)\mathcal{C}_{\mathbf{K}(\mu_Y)}.$$

**Proof.** The assertion concerning the mixture is clear. For the other one, following (3.1), it suffices to see that  $\mathbf{K}(\mu_{X.Y}) = \mathbf{K}(\mu_X) \star_{\mathcal{T}} \mathbf{K}(\mu_Y)$ . Observe that for any measure  $\nu$  on  $\mathbb{C}$  (such that  $0 < |X_\mu| < +\infty$ ),

$$\frac{\mathbb{E}(e^{ix \text{arg}(X_\mu)} |X_\mu|)}{\mathbb{E}(|X_\mu|)} = \int_0^{2\pi} e^{ix\theta} \frac{m_{X_\mu}(\theta)}{\mathbb{E}(|X_\mu|)} d\mathbb{P}_{\text{arg}(X_\mu)}(\theta).$$

In other words, the Fourier transform of  $\mathbf{K}(\mu)$  at position  $x$  is given by  $\frac{\mathbb{E}(e^{ix \text{arg}(X_\mu)} |X_\mu|)}{\mathbb{E}(|X_\mu|)}$ . This permits to see that the Fourier transform of  $\mathbf{K}(\mu_{X.Y})$  and of  $\mathbf{K}(\mu_X) \star_{\mathcal{T}} \mathbf{K}(\mu_Y)$  are the same.  $\square$

**Remark 4.5.** The convex  $\mathcal{C}_{\mathbf{K}(\mu)}$  characterises  $\mathbf{K}(\mu)$  but not  $\mu$ . For example the two following measures  $\mu_1 = \frac{1}{3}(\delta(1) + \delta(e^{2i\pi/3}) + \delta(e^{4i\pi/3}))$  and  $\mu_2 = \frac{1}{3}(\frac{1}{2}\delta(\frac{1}{2}) + \frac{1}{2}\delta(\frac{3}{2}) + \delta(e^{2i\pi/3}) + \delta(e^{4i\pi/3}))$  satisfies  $\mathbf{K}(\mu_1) = \mathbf{K}(\mu_2)$  and  $\mathcal{C}_{\mathbf{K}(\mu_i)}$  is an equilateral triangle. Every convex  $\mathcal{C}_\nu$  can therefore be seen as an equivalence class of measures over  $\mathbb{C}$ .

Now  $K\left(\mu_1 \star_{\mathcal{T}} \mu_1\right)$  represents a polygon with 6 sides, whereas  $K\left(\mu_1 \star_{\mathcal{T}} \mu_2\right)$  a polygon with 7 sides, even though  $K(\mu_1) = K(\mu_2)$ . Hence  $K(\mu_1 \star \mu_2)$  is not a function of  $K(\mu_1)$  and  $K(\mu_2)$ , and then the convolution of measures in  $\mathbb{C}$  can not be turned into a nice operation on convexes.

## 5 Some models of random convexes

In this part, we consider the problem of finding natural distributions on the convexes of the plane. We first recall some classical considerations on models of random convex polygons, and highlight the main problems for generating random paths that are also convex polygons, i.e. ascertaining that the paths are closed and that the vertices are in a convex position. In a second part, we take advantage of the representation of convexes by measures in  $\mathcal{M}_{\mathcal{T}}^0$  to present models for the generation of smooth convexes, based on the Fourier development of their associated measure.

### 5.1 Polygonal convexes

#### 5.1.1 Voronoï diagrams

Given a finite sequence of points in the plane  $\{P_k\}$ , one can define a Voronoï diagram, namely a partition of the plane into convex polygons. The polygon associated to a point  $P_i$  is the set of points  $\{Q, \forall P_k \neq P_i, d(P_k, Q) \geq d(P_i, Q)\}$ . Such polygons are finite intersections of half-planes, and therefore convex. As a consequence, every process generating points in the plane (for example an homogeneous Poisson process in the plane) induces a model for generating random convex polygons. In this particular case, many parameters are known, such as the number of vertices, the perimeter and the area of the convex (see for example [10]).

#### 5.1.2 Reordering of closed polygon

Consider the problem of generating a convex polygon by specifying a finite set of vectors representing its edges. We present here a simple way to ensure that the resulting path is also closed. Let  $\mu$  be a distribution on  $\mathbb{C}$  whose support is not reduced to a point, and for some  $n \geq 2$ , let  $(X_i, i = 1, \dots, n)$  be  $n$  i.i.d. r.v. distributed according to  $\mu$ . Let

$$W_i = X_{(i \bmod n)+1} - X_i, \quad 1 \leq i \leq n$$

be the sequence of increments. Naturally,  $\sum_{i=1}^n W_i = 0$ . Let  $(\hat{W}_i, 1 \leq i \leq n)$  be the sequence  $(W_i, 1 \leq i \leq n)$  sorted according to the argument. Let now  $S$  be defined as in (25), and  $\mathbf{C}_n$  defined as in Section 4.2. Further, let  $\nu$  be the distribution of  $X_2 - X_1$ , and  $\nu = K(\mu)$ .

The following analogous of Theorem 4.2 shows that  $\mathbf{C}_n$  converges in distribution to  $\mathcal{C}_\nu$  :

**Theorem 5.1.** *Assume that  $\mu$  is centred and admits a finite covariance matrix. We have  $d_H(\mathbf{C}_n/(n\mathbb{E}(|W|)), \mathcal{C}_\nu) \xrightarrow[n]{(d)} 0$ ; moreover (28) holds.*



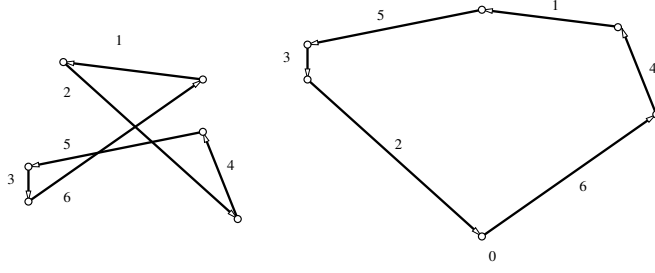


Figure 6: Reordering of an oriented polygon into a convex

**Proof.** The proof follows that of Theorem 4.2; the main change is due to the dependency of the  $W'_i$ s. Nevertheless, we get as in (29),  $N_n(\theta)/n \xrightarrow[n]{(proba.)} F_\nu(\theta)$ , since  $\mathbb{E}(N_n(\theta)/n) = F_\nu(\theta)$  and since  $\text{Var}(N_n(\theta)/n) \rightarrow 0$ . In the same way,

$$\frac{S(N_n(\theta))}{n\mathbb{E}(|W|)} = \sum_{i=1}^{N_n(\theta)} \frac{(X_i - X_{i-1})}{n\mathbb{E}(|W|)} \mathbf{1}_{\arg(X_i - X_{i-1}) \leq \theta} \quad (33)$$

$$\xrightarrow[n]{(proba.)} \frac{\mathbb{E}((X_i - X_{i-1}) \mathbf{1}_{\arg(X_i - X_{i-1}) \leq \theta})}{n\mathbb{E}(|W|)}. \quad (34)$$

Indeed, the convergence  $\mathbb{E}(S(N_n(\theta))/(n\mathbb{E}(|W|)))$  to the right-hand-side of (34) is simple, and a control of the second moment gives the convergence in distribution. The end of the proof follows the steps of the proof of Theorem 4.2.  $\square$

### 5.1.3 Convex through conditioning / convex by chance

Another natural probabilistic way to sample a convex is to take some i.i.d. r.v.  $W_0, \dots, W_{n-1}$  according to a distribution  $\mu$  (with support not included in a line), representing a set of vertices in the plane, and to condition  $(W_0, \dots, W_{n-1})$  to be convex. In this section, we define the set of all possible convex polygons as

$$\mathbf{C}_n = \{\mathbf{w} := (w_0, \dots, w_{n-1}) : \arg(w_{i+1 \bmod n} - w_i) \text{ forms an increasing sequence in } [0, 2\pi)\}.$$

Hence, if  $\mathbf{w}$  belongs to  $\mathbf{C}_n$ , then  $\mathbf{w}$  represents the list of vertices of a convex encountered when following its border in the counter-clockwise direction (with some conditions for  $w_0$ ).

The distribution  $\mu^{\otimes n}(\mathbf{C}_n)$  seems to be known for very few distributions  $\mu$ : for example Valtr [25, 24] answers the question when  $\mu$  is the uniform distribution in a fixed triangle or in a fixed parallelogram.

We open here a small parenthesis to explain the underlying difficulties.

Consider  $S_n := (w_0, \dots, w_n)$  a  $n$ -tuple of points in  $\mathbb{R}^2$ , not three of them being on the same line (this appears almost surely if  $\mu$  admits a density on an open set in  $\mathbb{R}^2$ ). For any

$0 \leq i < j < k \leq n-1$ , denote by  $a_{i,j,k}$  the area of the triangle  $(w_i, w_j, w_k)$ , and denote by  $s_{i,j,k} = \text{sign}(a_{i,j,k})$ . Recall that in the case where  $s_i = (x_i, y_i)$  for any  $i$ ,

$$a_{i,j,k} = \frac{1}{2}(x_i y_j + x_j y_k + x_k y_i - y_i x_j - y_j x_k - y_k x_i). \quad (35)$$

The sequence  $(s_{i,j,k}, 0 \leq i < j < k \leq n-1)$  is called the *chirotope* of  $S_n$ , and an equivalence class for the chirotope, is called an *order type*. The sequence  $S_n$  forms a convex, if all  $s_{i,j,k}$  have the same sign. It is known that some order types are empty, and also that to decide if an order type is not empty, is a *NP*-complete problem (see Knuth [16, Section 6]).

It turns out that some computations can be done in the Gaussian case: the joint law of the  $A'_{i,j,k}$ s can be computed, and expressed through their join Laplace transform.

Let  $(W_j = (X_j, Y_j), j = 0, \dots, n-1)$  form a family of i.i.d. r.v., where the  $X_i$  and  $Y_i$  are independent, and follow the Gaussian distribution with mean 0 and variance 1. Denote by  $A_{i,j,k}$  the area of  $(W_i, W_j, W_k)$ . Now, the Laplace transform of the sequence  $(A_{i,j,k}, 0 \leq i < j < k \leq n-1)$  is

$$\Phi(\lambda_{i,j,k}, 0 \leq i < j < k \leq n-1) := \mathbb{E} \left( \exp \left( \sum_{0 \leq i < j < k \leq n-1} \lambda_{i,j,k} A_{i,j,k} \right) \right)$$

converges in a neighbourhood of the origin of  $\mathbb{R}^{\binom{n}{3}}$ , and can be computed. Indeed, the density of  $((X_i, Y_i), i = 0, \dots, n-1)$  is given by  $\exp(-\sum_{i=0}^n (x_i^2 + y_i^2)/2)/(2\pi)^n$  (on  $\mathbb{R}^{2n}$ ); if  $(G_0, \dots, G_{n-1})$  is a Gaussian vector with mean 0 and covariance matrix  $\Gamma$  (that is  $\text{cov}(G_i, G_j) = \Gamma_{i,j}$ ), then its density is

$$f_\Gamma(x_0, \dots, x_{n-1}) = \frac{1}{(2\pi)^{n/2} |\Gamma|^{1/2}} \exp \left( -\frac{1}{2} \mathbf{x}^t \Gamma^{-1} \mathbf{x} \right)$$

where  $\mathbf{x} = {}^t(x_0, \dots, x_{n-1})$  (and then  $\int_{\mathbb{R}^n} f_\Gamma = 1$ ). This can be used to compute  $\Phi$ :

$$\int_{\mathbb{R}^{2n}} \exp \left( -\sum_{i=0}^{n-1} \frac{x_i^2 + y_i^2}{2} + \sum_{0 \leq i < j < k \leq n-1} \frac{\lambda_{i,j,k}}{2} (x_i y_j + x_j y_k + x_k y_i - y_i x_j - y_j x_k - y_k x_i) \right) \prod_{i=0}^{n-1} \frac{dx_i dy_i}{2\pi}.$$

Let us rewrite the polynomial in  $x_i, y_i$  in the exponential under the form

$$-\frac{1}{2} \left( \sum_{i=0}^{n-1} x_i^2 + y_i^2 + \sum_{i < j} (x_i y_j - y_i x_j) \bar{\lambda}_{i,j} \right)$$

where for  $i < j$ ,  $\bar{\lambda}_{i,j} := \sum_a \lambda_{i,j,a} + \lambda_{a,i,j} - \lambda_{i,a,j}$ , and  $\bar{\lambda}_{j,i} = -\bar{\lambda}_{i,j}$ ,  $\bar{\lambda}_{i,i} = 0$  where by convention  $\lambda_{a,b,c}$  equal 0 if  $0 \leq a < b < c \leq n-1$  does not hold. Therefore, the polynomial in the exponential is  $-\frac{1}{2} \mathbf{z}^t \Lambda \mathbf{z}$  for  $\mathbf{z}$ , the column vector with coordinates  $x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}$ , and where

$$\Lambda := \begin{bmatrix} Id_n & (-\bar{\lambda}_{i,j})_{i,j} \\ (\bar{\lambda}_{i,j})_{i,j} & Id_n \end{bmatrix}.$$

Since the matrix  $(\bar{\lambda}_{i,j})_{i,j}$  is antisymmetric,  $\Lambda$  is symmetric, and therefore, we get

$$\mathbb{E} \left( \exp \left( \sum_{0 \leq i < j < k \leq n-1} \lambda_{i,j,k} A_{i,j,k} \right) \right) = |\det(\Lambda)|^{-1/2},$$

using the considerations explained above concerning the Gaussian multivariate distribution.

**Remark 5.2.** As remarked by Andrea Sportiello in a private communication, the determinant of  $\Delta$  is always a square of a polynomial in the coefficients  $\lambda_{i,j}$ . The reason is that  $\Lambda$  and  $\Lambda' = \begin{bmatrix} -Id_n & 0 \\ 0 & Id_n \end{bmatrix} \Lambda$  have the same determinant (up to factor  $(-1)^n$ ). But  $\Lambda'$  is a skew matrix, and then, its determinant is the square of its Pfaffian, which is indeed a polynomial on its coefficients.

Hence, the joint law of the area of all triangles is known, but indirectly, via its Laplace transform (the Gaussian distribution is probably the simplest non trivial measure for which this computation is possible). The question of the emptiness of an order type  $S = (s_{i,j,k}, i < j < k)$  can be translated in term of the support of the measure: do we have  $\mathbb{P}(\text{sign}(A_{i,j,k}) = s_{i,j,k}) > 0$ ? According to the result of Knuth recalled above, extracting this information from the Laplace transform is necessarily a difficult task. If  $n = 3$ , only one triangle is present; the Laplace transform is then  $1/(1 - 3\lambda_{0,1,2}^2/4)$  revealing a Gamma distribution for the area; when  $n = 4$ , the Laplace transform is much more complex: this is due to the fact that in a standard position  $A_{1,2,4} + A_{2,3,4} + A_{1,4,3} = A_{1,2,3}$ . We were unable, even in the case  $n = 4$ , to find a better description of the distribution of  $(A_{i,j,k}, i < j < k)$  than their Laplace transform.

To learn more on order types, and on related questions, we send the interested reader to Goodman & al. [12] and to Aichholzer & al. [1], and references therein.

## 5.2 Generation of smooth random convexes

This part is mainly prospective. We provide some methods to generate random smooth convexes in  $\text{Conv}(1)$ . This is done from the generation of random “smooth” measures in  $\mathcal{M}_{\mathcal{T}}^0$  via their Fourier coefficients. Remains the question of finding natural distributions for convexes.

According to Szegő’s theorem [23], if a trigonometric polynomial  $P : \mathcal{T} \rightarrow \mathbb{R}^+$  admits only non-negative values, then there exists another trigonometric polynomial  $D$  such that :

$$\forall t \in \mathcal{T}, \quad P(t) = |D(e^{it})|^2$$

Moreover,  $D$  is unique up to multiplication by a complex of modulus 1. If we consider the Fourier expansion  $D(e^{it}) = \sum_{n \geq 0} \rho_n e^{i\theta_n} e^{int}$ , for some finite sequences of real numbers  $(\rho_n), (\theta_n)$ ,

the modulus of  $D$  is equal to :

$$|D(e^{it})|^2 = A_0 + \sum_{n \geq 1} A_n \cos(nt) + B_n \sin(nt) \quad \forall n \geq 1, \begin{cases} A_0 = \sum_{k \geq 0} \rho_k^2 \\ A_n = 2 \sum_{k \geq 0} \rho_{k+n} \rho_k \cos(\theta_k - \theta_{k+n}) \\ B_n = 2 \sum_{k \geq 0} \rho_{k+n} \rho_k \sin(\theta_k - \theta_{k+n}). \end{cases}$$

Hence, the polynomial  $P$  is the density of a measure  $\mu \in \mathcal{M}_{\mathcal{T}}^0$  iff the sequences  $(A_n)$  and  $(B_n)$  satisfy (i) the perimeter condition ( $A_0 = \frac{1}{2\pi}$ , ensuring that  $\mu$  is a probability measure) and (ii) the closed path condition ( $A_1 = B_1 = 0$ , ensuring that  $\int_0^{2\pi} e^{ix} d\mu(x) = 0$ ).

### 5.2.1 Generation of convex sets via their Fourier coefficients

In order to generate a random pair  $\mathcal{P} := ((\rho_k, k \geq 0), (\theta_k, k \geq 0))$  satisfying both conditions, two possibilities are open, depending on which condition should be satisfied first.

To satisfy the closed path condition first, namely  $A_1 = B_1 = 0$ , it suffices to generate  $\rho_j$  and  $\theta_j$  for  $j \geq 1$  at random, then take  $\rho_0$  and  $\theta_0$  such that :

$$\rho_0 \rho_1 e^{i(\theta_0 - \theta_1)} = - \sum_{k \geq 1} \rho_{k+1} \rho_k e^{i(\theta_k - \theta_{k+1})}.$$

This is well defined if  $\sum_{k \geq 1} \rho_{k+1} \rho_k$  are 0 for all but a finite number of them. In this case, it is unlikely that the perimeter condition be satisfied. In order to get a convex with perimeter 1, a normalisation step must be applied, which consists in dividing each  $\rho_n$  by  $\sqrt{\sum_{k \geq 0} \rho_k^2}$ .

Szegő's theorem ensures that the set of measures induced by this process has full support over  $\mathcal{M}_{\mathcal{T}}^0$ : indeed, each measures in  $\mathcal{M}_{\mathcal{T}}^0$  can be weakly approached by a sequence of distributions with strictly positive density; these ones can be in turn approached by a sequence of positive trigonometric polynomials, and Szegő's Theorem gives a representation of these polynomials. The results of such a generation can be seen on figure 7.

Another solution consists in ensuring first the perimeter condition, which comes down to producing  $(\rho_k, k \geq 0)$  such that  $\sum_{k \geq 0} \rho_k^2 = \frac{1}{2\pi}$ . To this purpose, it suffices to generate random reals  $r_j$  in  $[0, 1]$ , and set :

$$\rho_k^2 = \frac{1}{2\pi} r_k \prod_{j=0}^{k-1} (1 - r_j).$$

This is well defined if  $\prod_k (1 - r_k)$  converges to 0 when  $k$  goes to infinity (for example, taking i.i.d.  $r_j$ 's under  $\text{uniform}[0, 1]$  does the job). From here, satisfying  $A_1 = 0$  and  $B_1 = 0$  by a right choice of  $\theta$ 's can become more difficult, and even impossible, for example if  $\rho_0 = \rho_1 > 0$  and all other  $\rho_i$ 's are 0.

Nevertheless, it is possible to generate  $\mathcal{P}$  satisfying all the constraints at once. Choose (at random or not) a subset  $F$  of  $\mathbb{N}$  such that if  $i \in F$ , then  $i + 1 \notin F$ , and a sequence  $x_k$  such that

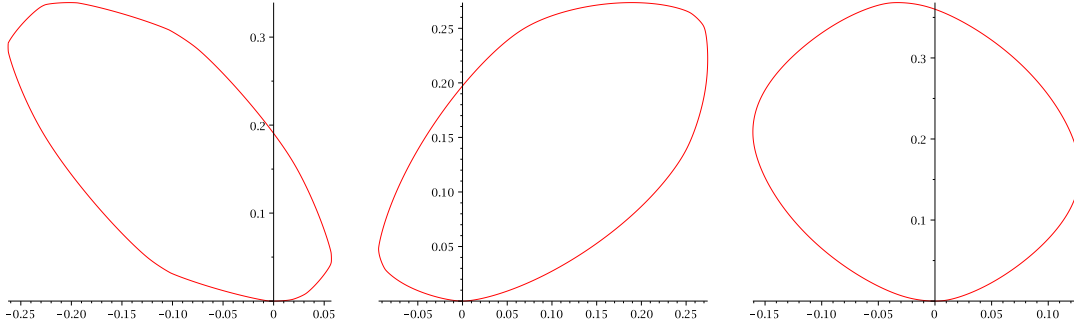


Figure 7: Examples of random convex sets generated from trigonometric polynomials containing 25 non-zero coefficients (with  $\rho_j \sim \text{uniform}[0; 1]$ , and  $\theta_j \sim \text{uniform}[0; 2\pi]$ ), all these r.v. being taken independently.

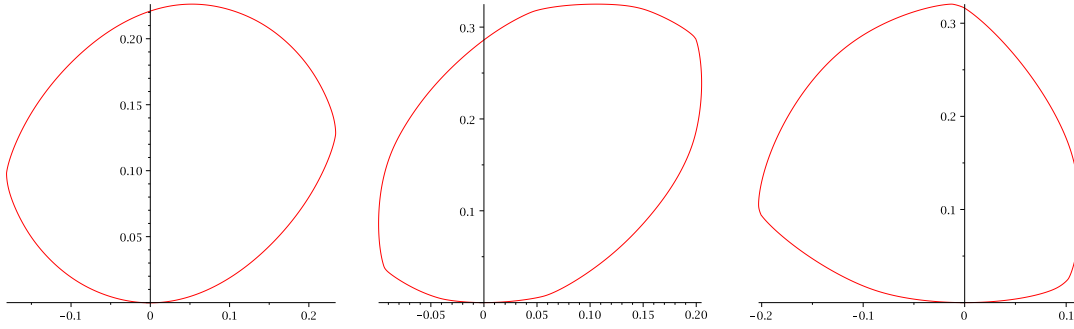


Figure 8: Examples of random convex sets generated from polynomials containing 12 non-zero coefficients with sparse coefficients (the indices of the non-null Fourier coefficients of  $F$  are selected with a Bernoulli of parameter  $\frac{1}{2}$ , and  $\rho_j \sim \text{uniform}[0; 1]$ , and  $\theta_j \sim \text{uniform}[0; 2\pi]$ , all these r.v. being taken independently).

$\sum_{k \geq 0} x_k^2 = \frac{1}{2\pi}$  as above. Now, let  $n_j$  be the  $j$ -th smallest element in  $F$ , with the convention that the smallest is  $n_0$ . Define the sequence  $(\rho_k)$  by :

$$\rho_{n_j} = r_j, \quad \rho_k = 0 \text{ otherwise}$$

Thanks to the definition of  $F$ , the equations  $A_1 = B_1 = 0$  will be satisfied for any choice of the  $(\theta_k)$ . Examples of convex sets generated this way appear on figure 8.

### 5.2.2 Generation of convex sets of given area

Consider the problem of generating a convex in  $\text{Conv}(1)$  possessing a given area  $\alpha \in [0, \frac{1}{2\pi^2}]$ . Let  $\beta$  such that  $\alpha = \frac{1}{4\pi} - \frac{\pi}{2}\beta$ . In order to obtain a convex set with area  $\alpha$ , it is necessary that

its Fourier coefficients verify :

$$\sum_{k \geq 2} \frac{a_k^2 + b_k^2}{k^2 - 1} = \beta$$

As in the previous section, we consider a sequence of numbers  $(r_j)$  in  $[0, 1)$  for  $j \geq 2$ , such that  $\prod_{j \geq 2} (1 - r_j) = 0$ , and define positive reals  $(c_k)$  such that :

$$\frac{c_k^2}{k^2 - 1} = \beta r_k \prod_{j=2}^{k-1} (1 - r_j)$$

Let  $(\theta_k, k \geq 2)$  be a sequence of real numbers in  $[0, 2\pi)$ . Then the Fourier coefficients of the associated measure can be computed as follows :

$$a_k = \cos(\theta_k) c_k, \quad b_k = \sin(\theta_k) c_k.$$

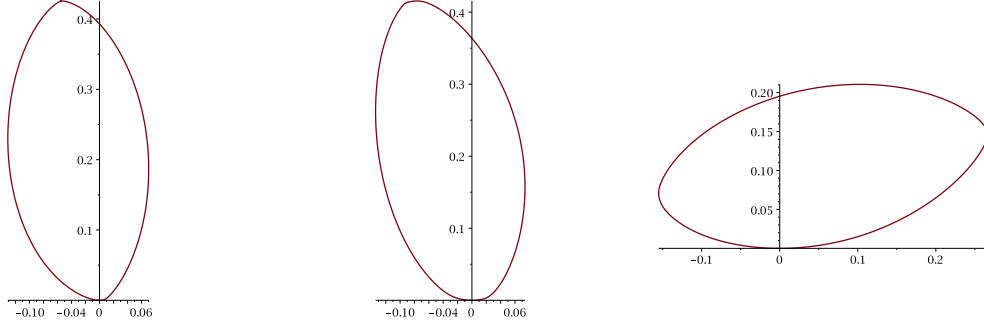


Figure 9: Examples of random convex sets of perimeter 1 generated such that their area be equal to  $\frac{1}{4\pi} - \frac{\pi}{2} \times 0.01$  (the polynomials possess 20 non-null coefficients,  $\rho_j \sim \text{uniform}[0; 1]$ , and  $\theta_j \sim \text{uniform}[0; 2\pi]$ , all these r.v. being taken independently.)

It is still possible to take  $a_1 = 0$ ,  $b_1 = 0$  and  $a_0 = 1/(2\pi)$ , but since we didn't use Szegő's theorem, the standard Fourier series associated to the  $a_i$ 's and  $b_i$ 's is unlikely to be a positive function. From here, it suffices to reject all series with a negative minimum. The results of such a generation appear on figure 9. Experiments show that the rejection rate is very high, and that it is very difficult to generate convex sets with  $\beta > 0.01$  (the theoretical maximum being  $\frac{1}{2\pi^2} \approx 0.05$ ).

## 6 Appendix

### 6.1 Proof of Theorem 2.7

First, the second assertion (2) is a consequence of the first one. The proof of (1) starts with the proof of the convergence of the FDD of  $W_n$ . Let  $\theta_0 := 0 \leq \theta_1 < \theta_2 < \dots < \theta_\kappa = 2\pi$  for some

$\kappa \geq 1$  be fixed. In the sequel, for any function (random or not)  $L$  indexed by  $\theta$ ,  $\Delta L(\theta_j)$  will stand for  $L(\theta_j) - L(\theta_{j-1})$ . For any  $\ell \leq \kappa$

$$W_n(\theta_\ell) = \sqrt{n} \sum_{j=0}^{\ell} \Delta [Z_n(N_n(\theta_j)) - Z_\mu(F_\mu(\theta_j))]. \quad (36)$$

where by convention  $Z_n(N_n(\theta_{-1})) = Z_\mu(F_\mu(\theta_{-1})) = 0$ . The convergence of the FDD of  $W_n$  follows those of  $(\sqrt{n} \Delta [Z_n(N_n(\theta_j)) - Z_\mu(F_\mu(\theta_j))], 0 \leq i \leq \kappa)$ . Notice that

$$\Delta Z_\mu(F_\mu(\theta_j)) = \mathbb{E}(\exp(iX) 1_{\theta_{j-1} < X \leq \theta_j}). \quad (37)$$

If for some  $j$ ,  $\theta_{j-1}$  and  $\theta_j$  are chosen in such a way that  $\Delta F_\mu(\theta_j) = 0$  then the  $j$ th increment in (36) is 0 almost surely (this is the case for the 0th increment if  $\mu(\{0\}) = 0$ ). We now discuss the asymptotic behaviour of the other increments : let  $J = \{j \in \{0, \dots, \kappa\} : \Delta F_\mu(\theta_j) \neq 0\}$ .

Let  $(n_j, j \in J)$  be some fixed integers summing to  $n$ . Denote by  $\mu_{\theta_{j-1}, \theta_j}$  the law of  $X_\mu$  conditioned by  $\{\theta_{j-1} < X_\mu \leq \theta_j\}$ . Conditionally on  $(N_n(\theta_j) = n_j, j \in J)$ , the variables  $\Delta Z_n(N_n(\theta_j)), j \in J$  are independent. The law of  $\Delta Z_n(N_n(\theta_j))$  is that of a sum of  $n_j - n_{j-1}$  i.i.d. copies of variables under  $\mu_{\theta_{j-1}, \theta_j}$ , denoted from now on  $(X_{\theta_{j-1}, \theta_j}(k), k \geq 1)$ :

$$\begin{aligned} \mathbb{E}(\Delta Z_n(N_n(\theta_j)) | N_n(\theta_l) = n_l, l \in J) &= n^{-1} \mathbb{E} \left( \sum_{m=1}^{n_j - n_{j-1}} e^{iX_{\theta_{j-1}, \theta_j}(m)} \right) \\ &= \frac{(n_j - n_{j-1})}{n} \frac{\Delta Z_\mu(F_\mu(\theta_j))}{\Delta F_\mu(\theta_j)}. \end{aligned}$$

Since  $(\Delta N_n(\theta_j), j \in J) \sim \text{Multinomial}(n, (\Delta F_\mu(\theta_j), j \in J))$ ,

$$\left( \frac{\Delta N_n(\theta_j) - n \Delta F_\mu(\theta_j)}{\sqrt{n}}, j \in J \right) \xrightarrow[n]{(d)} (N_j, j \in J) \quad (38)$$

where  $(N_j, j \in J)$  is a centred Gaussian vector with covariance function

$$\text{cov}(N_k, N_j) = -\Delta F_\mu(\theta_k) \cdot \Delta F_\mu(\theta_j),$$

formula valid for any  $0 \leq k, l \leq \kappa$ . Putting together the previous considerations, we have, conditioning first on the  $N_n(\theta_j)$ 's, and then integrating on the distribution of this variable,

$$\Delta W_n(\theta_j) = \sum_{l=1}^{\Delta N_n(\theta_j)} \frac{e^{iX_{\theta_{j-1}, \theta_j}(l)} - \mathbb{E}(e^{iX_{\theta_{j-1}, \theta_j}})}{\sqrt{n}} + \left( \frac{\Delta N_n(\theta_j) - n \Delta F_\mu(\theta_j)}{\sqrt{n}} \right) \mathbb{E}(e^{iX_{\theta_{j-1}, \theta_j}}) \quad (39)$$

Using (38) and the central limit theorem, we then get that

$$(\pi \Delta W_n(\theta_j), 0 \leq j \leq \kappa) \xrightarrow[n]{(d)} \sqrt{\Delta F_\mu(\theta_j)} \tilde{N}_j + N_j \begin{bmatrix} \mathbb{E}(\cos(X_{\theta_{j-1}, \theta_j})) \\ \mathbb{E}(\sin(X_{\theta_{j-1}, \theta_j})) \end{bmatrix}, \quad (40)$$

where the variables  $N_j, \tilde{N}_j, j \leq \kappa$  are independent, and the variables  $\tilde{N}_j$  are centred Gaussian variables with covariance matrix that of the vector  $\begin{bmatrix} \cos(X_{\theta_{j-1}, \theta_j}) \\ \sin(X_{\theta_{j-1}, \theta_j}) \end{bmatrix}$ .

It remains to show the tightness of the sequence  $(W_n, n \geq 0)$  in  $D[0, 1]$ . A criterion for the tightness in  $D[0, 2\pi]$  can be found in Billingsley [5, Thm. 13.2]: a sequence of processes  $(W_n, n \geq 1)$  with values in  $D[0, 2\pi]$  is tight if, for any  $\varepsilon \in (0, 1)$ , there exists  $\delta > 0, N > 0$  such that

$$\lim_{\delta \rightarrow 0} \limsup_n \mathbb{P}(\omega'(W_n, \delta) \geq \varepsilon) = 0$$

where  $\omega'(f, \delta) = \inf_{(t_i)} \max_i \sup_{s, t \in [t_{i-1}, t_i]} |f(s) - f(t)|$ , and the partitions  $(t_i)$  range over all partitions of the form  $0 = t_0 < t_1 < \dots < t_n \leq 2\pi$  with  $\min\{t_i - t_{i-1}, 1 \leq i \leq n\} \geq \delta$ .

Since only the tightness in  $D[0, 2\pi]$  interests us, we will focus on  $\Re(W)$  (since the imaginary part can be treated likewise, and since the tightnesses of both  $\Re(W)$  and  $\Im(W)$  implies that of  $W$ ). For the sake of brevity, in the sequel, we will use  $W$  instead of  $\Re(W)$ .

The first step in our proof consists in comparing our current model formed by a set  $\{X_1, \dots, X_n\}$  of  $n$  i.i.d. copies of  $X_\mu$  denoted from now on by  $\mathbb{P}_n$ , with a Poisson point process  $P_n$  on  $[0, 2\pi]$  with intensity  $n\mu$ , denoted by  $\mathbb{P}_{P_n}$ . Conditional on  $\#P_n = k$ , the  $k$  points  $P_n := \{Y_1, \dots, Y_k\}$  are i.i.d. and have distribution  $\mu$ , and then  $\mathbb{P}_{P_n}(\cdot | \#P = n) = \mathbb{P}_n$ . The Poisson point process is naturally equipped with a filtration  $\sigma := \{\sigma_t = \sigma(\{P \cap [0, t]\}), t \in [0, 2\pi]\}$ .

We are here working under  $\mathbb{P}_{P_n}$ , and we let  $N(\theta) = \#P_n \cap [0, \theta]$ ; notice that under  $\mathbb{P}_n$ ,  $N$  and  $N_n$  coincide.

Before starting, recall that if  $N \sim \text{Poisson}(a)$ , for any positive  $\lambda$ ,

$$\mathbb{P}(N \geq x) = \mathbb{P}(e^{\lambda N} \geq e^{\lambda x}) \leq \mathbb{E}(e^{\lambda N - \lambda x}) = e^{-a + ae^{\lambda} - \lambda x} \quad (41)$$

$$\mathbb{P}(N \leq x) = \mathbb{P}(e^{-\lambda N} \geq e^{-\lambda x}) \leq \mathbb{E}(e^{-\lambda N + \lambda x}) = e^{-a + ae^{-\lambda} + \lambda x}, \quad (42)$$

and optimising the right hand side on  $\lambda > 0$  gives quite good inequalities.

We will show the tightness of  $W$  under  $\mathbb{P}_{P_n}$  first. Before doing this, we explain why it implies the same result under  $\mathbb{P}_n$ : Let  $m = \inf\{x \in [0, 2\pi], F_\mu(x) \geq 1/2\}$  the median of  $\mu$ . We will see that the tightness under  $\mathbb{P}_{P_n}$  implies that the sequence of processes  $W$  under  $\mathbb{P}_n$  is tight in  $D[0, m]$  (the same proof works on  $D[m, 2\pi]$  by a time reversing argument). We claim that for any event  $\sigma_m$  measurable,

$$\mathbb{P}_n(A) = \mathbb{P}_{P_n}(A | \#P = n) \leq c \mathbb{P}_{P_n}(A) \quad (43)$$

for a constant  $c$  independent on  $n$  and of  $A$  (but which depends on  $\mu$ ). This in hand, the tightness



under  $\mathbb{P}_{P_n}$  of  $W$  on  $D[0, m]$  implies that under  $\mathbb{P}_n$ . Let us prove (43). We have

$$\begin{aligned}\mathbb{P}_{P_n}(A \mid \#P = n) &= \sum_k \frac{\mathbb{P}_{P_n}(A, \#(P \cap [0, m]) = k) \mathbb{P}(\#P \cap [m, 2\pi] = n - k)}{\mathbb{P}(\#P = n)} \\ &\leq \sum_k \mathbb{P}_{P_n}(A, \#(P \cap [0, m]) = k) \sup_{k'} \frac{\mathbb{P}(\#P \cap [m, 2\pi] = n - k')}{\mathbb{P}(\#P = n)} \\ &\leq c \mathbb{P}_{P_n}(A)\end{aligned}$$

where  $c = \sup_{n \geq 1} \sup_{k'} \frac{\mathbb{P}(\#P \cap [m, 2\pi] = n - k')}{\mathbb{P}(\#P = n)}$ , which is indeed finite since  $\mathbb{P}(\#P = n) \sim (2\pi n)^{-1/2}$ , and since  $\#P \cap [m, 2\pi] \sim \text{Poisson}(n/2)$ , and then the probability that its value is  $k$  is bounded above by some  $d/\sqrt{n}$  according to Petrov [18, Thm. 7 p. 48].

Let  $A_\mu = \{x \in [0, 2\pi], \mu(\{x\}) > 0\}$  be the set of positions of the atoms of  $\mu$ . We now decompose  $\mu = \mu|_{A_\mu} + \mu|_{\mathbb{C}A_\mu}$ ; under  $\mathbb{P}_n$  as well as under  $\mathbb{P}_{P_n}$ , the process  $W = W|_{A_\mu} + W|_{\mathbb{C}A_\mu}$  decomposes according to these measures clearly (setting  $N|_{A_\mu}(\theta) = \#P \cap [0, \theta] \cap A_\mu$ ,  $Z|_{A_\mu}(N|_{A_\mu}(\theta)) = \sum_{j=1}^N e^{i\hat{X}_j} 1_{\hat{X}_j \in A_\mu}$ , etc). The fluctuation of  $W = W|_{A_\mu} + W|_{\mathbb{C}A_\mu}$  are then bounded by the sum of the fluctuations of both processes  $W|_{A_\mu}$  and  $W|_{\mathbb{C}A_\mu}$ . It is then sufficient to show the tightness for a purely atomic measure  $\mu$ , and for a measure having no atom  $\mu$ .

To start, we assume that  $\mu$  is purely atomic. Take then some (small)  $\eta \in (0, 1)$ ,  $\varepsilon > 0$ ; we will show that one can find a finite partition  $(t_i, i \in I)$  of  $[0, 2\pi]$  and a  $\delta \in (0, 1)$  such that

$$\limsup_n \mathbb{P}_n(\omega'(W_n, \delta) \geq \varepsilon) \leq \eta, \quad (44)$$

which is sufficient for our purpose. In fact we will establish (44) under  $\mathbb{P}_{P_n}$  instead, on  $[0, m]$  and then on  $[m, 2\pi]$ , since we saw that this was sufficient (replacing  $\eta$  by  $c\eta$  in (44), suffices too).

Now, let  $A_\mu^{\geq a} := \{x \in A_\mu : \mu(\{x\}) \geq a\}$ . Clearly  $\#A_\mu^{\geq a} \leq 1/a$  and  $[0, 2\pi] \setminus A_\mu^{\geq a}$  forms a finite union of open connected intervals  $(O_x, x \in G)$ , with extremities  $(t'_i, i \in I)$ . The intervals  $(O_x, x \in G)$  can be further cut as follows:

- do nothing to those such that  $\mu(O_x) < 2a$ ,
- those such that  $\mu(O_x) > 2a$  are further split. Since they contain no atom with mass  $> a$ , they can be split into smaller intervals having all their weights in  $[a, 2a]$  except for at most one (in each interval  $O_x$  which may have a weight smaller than  $a$ ).

Once all these splitting have been done, a list of at most  $3/a$  intervals are obtained, all of them having a weight smaller than  $2a$ . Name  $G_a = (O_x, x \in I_a)$  the collection of obtained open intervals, index by  $I_a$ , and by  $(t_i^a, i \geq 0)$  the partitions obtained. Clearly

$$M_a := \max_{i \in I_a} \mathbb{E}(\cos(X_\mu)^2 1_{X_\mu \in O_i}) \leq M'_a := 2a.$$

In the sequel we take  $a = \varepsilon^3$  and consider a unique interval  $O_x = (\theta_{j-1}, \theta_j) \in G_a$ , in which case we have  $M_{\varepsilon^3} \leq 2\varepsilon^3$ . Under  $\mathbb{P}_{P_n}$ ,  $\mathcal{P}(n\mu\{\theta\}) := \#P_n \cap \{\theta\}$  has distribution  $\text{Poisson}(n\mu(\{\theta\}))$ , the

variables corresponding to different points being independent. Following (39), under  $\mathbb{P}_{P_n}$ , we get

$$\Delta W_n(\theta_j) = \sqrt{n} \sum_{\substack{\theta \in A_\mu \\ \theta_{j-1} \leq \theta < \theta_j}} \left( \frac{\mathcal{P}(n\mu\{\theta\})}{n} - \mu(\{\theta\}) \right) \cos(\theta). \quad (45)$$

These centred random variables can be controlled as usual Poisson r.v. (as recalled above). On the first hand, we have

$$\mathbb{P}(\Delta W_n(\theta_j) \geq \varepsilon) = \mathbb{P} \left( \sum_{\theta} \mathcal{P}(n\mu\{\theta\}) \cos(\theta) \geq y \right) \quad (46)$$

with the set of summation is the same as before, but from now on, suppressed and

$$y = \varepsilon\sqrt{n} + n\mathbb{E}(\cos(X)1_{X \in A_\mu, \theta_{j-1} < X \leq \theta_j}). \quad (47)$$

Writing  $\mathbb{P}(\sum_{\theta} \mathcal{P}(n\mu\{\theta\}) \cos(\theta) \geq y) \leq \inf_{\lambda > 0} e^{-\lambda y} \prod_{\theta} \mathbb{E}(e^{\lambda \cos(\theta) \mathcal{P}(n\mu\{\theta\})})$  one has

$$\mathbb{P}(\Delta W_n(\theta_j) \geq \varepsilon) \leq \inf_{\lambda > 0} \exp \left( - \sum_{\theta} n\mu\{\theta\} + \sum_{\theta} n\mu\{\theta\} e^{\lambda \cos(\theta)} - \lambda y \right)$$

To get a bound, we will take  $\lambda = \varepsilon/(\sqrt{n}M'_{\varepsilon_3})$ . This allows one to bound  $e^{\lambda \cos(\theta)}$  by  $1 + \lambda \cos(\theta) + \lambda^2 \cos(\theta)^2$ , equality valid uniformly for any  $\theta$ , provided that  $n$  is large enough. Hence, for  $n$  large enough, replacing  $y$  by its value,

$$\begin{aligned} \mathbb{P}(\Delta W_n(\theta_j) \geq \varepsilon) &\leq \inf_{\lambda > 0} \exp \left( \lambda^2 n \mathbb{E}(\cos^2(\theta) 1_{\theta \in I_x}) - \lambda \varepsilon \sqrt{n} \right) \\ &\leq \inf_{\lambda > 0} \exp \left( \lambda^2 n M'_{\varepsilon_3} - \lambda \varepsilon \sqrt{n} \right) \\ &\leq \exp(-1/(4\varepsilon)) \end{aligned}$$

this last equality being obtained for  $\lambda = \varepsilon/(2M'_{\varepsilon_3}\sqrt{n})$ .

The proof for the control of  $\mathbb{P}(\Delta W_n(\theta_j) \leq -\varepsilon) \leq \inf_{\lambda > 0} \mathbb{E}(e^{-\lambda \Delta W_n(\theta_j) - \lambda \delta})$  for  $\delta > 0$  gives rise to the same estimates, except that the bound  $e^{\lambda \cos(\theta)}$  by  $1 - \lambda \cos(\theta) + \lambda^2 \cos(\theta)^2/4$  is taken to replace the other one, giving a bound  $\exp(-1/(2\varepsilon))$  at the end.

Now, we have to control the fluctuations, and not only the terminal value of the random walk. For this, we use Petrov [18, Thm.12 p50] which permits to control the first ones using the second ones.

The control of all intervals all together can be achieved using the union bound : since they are at most  $3/\varepsilon^3$  such intervals, by the union bound

$$\mathbb{P}_{P_n}(\sup_j \Delta W_n(\theta_j) \geq \varepsilon) \leq 3\varepsilon^{-3} e^{-1/(4\varepsilon)}.$$

This indeed goes to 0.

We now show the tightness of  $W$  under  $\mathbb{P}_{P_n}$  when  $\mu$  has no atom, and use the same reasoning as before: we work under  $\mathbb{P}_{P_n}$ , cut  $[0, 2\pi]$  under sub-intervals  $[t_{j-1}, t_j]'$ s, control the differences between starting and ending values on this intervals, since we saw that it was sufficient.

First we cut  $[0, 2\pi]$  into  $n$  (tiny) equal parts  $([2\pi(j-1)/n, 2\pi j/n], j = 1, \dots, n)$ . From (39)

$$W(2\pi j/n) - W(2\pi j'/n) = \sum_{l=j'+1}^j \Gamma_l + \Theta_l \quad (48)$$

where, under  $\mathbb{P}_{P_n}$ , denoting further  $\theta_j = 2\pi j/n$ ,

$$\begin{aligned} \Gamma_l &= \sum_{m=1}^{\mathcal{P}(n\Delta(F_\mu(\theta_l)))} \frac{\cos(X_{\theta_{j-1}, \theta_j}(m)) - \mathbb{E}(\cos(X_{\theta_{j-1}, \theta_j}))}{\sqrt{n}} \\ \Theta_l &= \frac{\mathcal{P}(n\Delta(F_\mu(\theta_l))) - n\Delta F_\mu(\theta_l)}{\sqrt{n}} \mathbb{E}(\cos(X_{\theta_{l-1}, \theta_l})) \end{aligned}$$

and  $\mathcal{P}(\lambda) \sim \text{Poisson}(\lambda)$  and the different Poisson r.v. appearing in all  $\Gamma_l$  are independent.

Let  $\varepsilon > 0$  be given, and  $N_{\varepsilon^3} = \lceil 1/\varepsilon^3 \rceil$ . Since  $\mu$  has a density, there exists some times  $t_0 = 0 < t_1, \dots < t_{N_\varepsilon} = 2\pi$  such that  $\mu([t_{i-1}, t_i]) \leq \varepsilon^3$ . We now control the fluctuations of  $W$  on the intervals  $[t_{j-1}, t_j]'$ s:

Write  $D_j := W(\frac{\lfloor 2\pi t_j n \rfloor}{n}) - W(\frac{\lfloor 2\pi t_{j-1} n \rfloor}{n})$  as a sum of variables  $\Gamma_l$  and  $\Theta_l$  as in (48). We use now standard exponential inequalities to control the deviations of  $D_j$ . The sum of the parameters of the Poisson falling in  $[t_{j-1}, t_j]$  is smaller than  $\varepsilon^3 n$  under  $\mathbb{P}_{P_n}$ . Therefore from (41),  $\mathbb{P}_{P_n}(N(t_j) - N(t_{j-1}) \geq 3\varepsilon^3 n) \leq e^{-c\varepsilon^3 n}$  for some positive  $c$ . The sum on  $\Gamma_l$  contributing to  $W(\frac{\lfloor 2\pi t_j n \rfloor}{n}) - W(\frac{\lfloor 2\pi t_{j-1} n \rfloor}{n})$  is then constituted by less than  $3\varepsilon^3 n$  contributions of the form  $\frac{\cos(X_{\theta_{j-1}, \theta_j}(m)) - \mathbb{E}(\cos(X_{\theta_{j-1}, \theta_j}))}{\sqrt{n}}$ , that are bounded random variables. By Hoeffding, conditionally on  $N(t_j) - N(t_{j-1}) \leq 3\varepsilon^3 n$ , the probability that this contribution is larger than  $\varepsilon$  goes to 0 with speed  $\exp(-c/\varepsilon)$  for some  $c > 0$ . The contribution of the sum on  $\Theta_l$  is then controlled as above, in the atomic case.  $\square$

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